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COMPLETE CONICS OF S_2 AND THEIR MODEL VARIETY $\Omega_5^{102}[27]$

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CONTENTS

This is a study of the aggregate of complete conics in the plane S_2 and their representation in S_{27} by the points of a model variety Ω of dimension 5 and order 102. Equivalence bases in each dimension on Ω are obtained and described, together with bases on certain important subvarieties of Ω, with a particular view to clarifying the enumerative geometry of complete conics. A number of important and interesting systems of complete conics are studied in some detail.

Introduction

The intention of this work is to record a detailed study of a particularly important geometric variable – the complete conic of S_2 – together with its subsidiary variables. This is the variable of which preliminary studies have been made independently by Severi and van der Waerden, mainly because it confronts the general theory of enumerative geometry with an almost ideal example of its applicability in practice. Briefly, a complete conic of S_2 is a couple composed of a conic-locus S and a conic-envelope E which are so related that if S is irreducible, then E is the envelope of tangents to S, while otherwise the couple (S, E) may be any proper specialization of a generic couple of the former type. We shall find that the aggregate of complete conics is an ∞^5-system whose standard model is an irreducible fivefold variety $\Omega = \Omega_5^{102}[27]$, and our study will be basically equivalent to a detailed exposition of geometry on this variety, including the classification of its sub-varieties of different dimensions into their separate systems of algebraic equivalence.

In what follows, we shall use the symbols **[S]** and **[W]**, for detailed references, when referring to the two most important accounts of the subject given by Severi and van der Waerden respectively. **[S]** is a substantial part (some 40 pages) of Severi (1940)†, a preliminary version of which had previously appeared in an earlier paper (Severi 1916) on the same general field of enumerative geometry. **[W]** is van der Waerden (1938); this is a paper of outstanding clarity and power, so much so that we shall not hesitate to follow its leads very closely indeed in the earlier parts of the exposition.

† I have not been able to make much use of the earlier theoretical part of Severi's memoir **[S]**, mainly for the reason that its arguments and assertions seemed to me to lack the necessary precision to carry conviction. This applied particularly to §§36–41, headed 'Digressione su alcune proprieta della base', in which, for example, Severi's statement and proof of his central lemma (§37) completely eluded my comprehension.

Notation

We use the symbol ${}^pV^n_d[r]$ to indicate that V is a variety of dimension d, order n and section genus p lying in projective space S_r; and we usually omit any letters in the symbol that are not of immediate interest. We also use the shorthand ${}^pF^n \equiv C^m(O_1^{k_1}, \ldots, O_s^{k_s})$ to indicate that the surface F is the projective model of a system of plane curves of order m having base points O_i of multiplicity k_i $(i = 1, \ldots, s)$. Other symbols in constant use throughout the paper are given in the following list.

$\widehat{S, E}$	a complete conic of S_2 with S as locus and E as envelope
Ω	the standard model $\Omega_5^{102}[27]$ of all $\widehat{S, E}$
δ, η	the varieties on Ω representing δ-conics and η-conics
$\delta\eta$	the intersection of δ and η, representing $\delta\eta$-conics
A_5, B_5	the spaces of conic-loci and conic-envelopes (of S_2) respectively
ϕ	the Veronese surface in A_5 which maps repeated lines of S_2
M	the cubic symmetroid M_4^3, locus of conic-planes of ϕ
a, b	conics on $\delta\eta$ representing $\delta\eta$-conics with a fixed vertex or a fixed axis, respectively
α	the plane of a conic a, representing δ-conics with a fixed vertex
β	the plane of a conic b, representing η-conics with a fixed axis
ρ	a surface ${}^2F^{12}$ on $\delta\eta$ mapping $\delta\eta$-conics with vertices on a fixed line
σ	another surface ${}^2F^{12}$ on $\delta\eta$ mapping $\delta\eta$-conics with axes through a fixed point
μ	condition on $\widehat{S, E}$ that S be apolar to a fixed conic-envelope of S_2
ν	condition on $\widehat{S, E}$ that E be apolar to a fixed conic-locus of S_2
$\bar{\mu}$	condition that S pass through a fixed point
$\bar{\nu}$	condition that E touch a fixed line
$\widehat{\mu\nu}$	condition on $\widehat{S, E}$ that it have a given point and line as pole and polar
$\overline{\mu\nu}$	condition that $\widehat{S, E}$ touch a given line at a given point
χ	a del Pezzo sextic surface, image on Ω of a trisecant plane of ϕ
τ	a rational scroll ${}^0R^5$, image on Ω of a tangent plane of ϕ
ω	a rational surface ${}^3F^{14}$, image on Ω of δ-conics with arms passing separately through two fixed points
κ	a rational scroll ${}^0R^6$, image on Ω of δ-conics whose arms correspond in a given harmonic homology of S_2
J	a threefold ${}^0R_3^9$, image on δ of δ-conics with vertices on a fixed line
$\lvert K \rvert$	the complete system of cubic primals of A_5 that pass through ϕ; the variety Ω is the projective model of this system

1. Severi's point-prime couples of A_5

Before embarking on the formal identification of complete conics as given in [W], which we shall take as our basis, we must indicate briefly the nature of Severi's prior definition as given in the two papers just mentioned.

If σ denotes the well known mapping of the conic-loci S of S_2 on the points P of a projective space A_5, then it is well known also that σ induces a dual mapping σ' of the conic-envelopes E of S_2 on the primes Π of A_5; in fact Π is the prime of A_5 whose points map, under σ, all the conic-

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loci S that are apolar (conjugate) to a given envelope E. Now let $M = M_4^3$ be the cubic primal of A_5 that maps under σ the line-pairs of S_2, and let ϕ be the Veronese surface – the double surface of M – that maps under σ the repeated lines of S_2. It appears, then, that if P is the image under σ of an *irreducible* conic-locus S, and if E is the envelope of tangents to S, then the image of E under σ' is the polar prime Π of P with respect to M. Severi then identifies the complete conics of S_2 by the aggregate of the couples (P, Π) of S_5 such that either Π is the polar prime of P, when this polar prime exists, or Π is the limit of such a polar prime when a variable point of A_5 approaches a point P of ϕ. This leads to a classification of complete conics identical with that which we describe in the next section.

2. The eight fundamental relations

In this section we follow closely the lines laid down in [**W**].

Let $A = (a_{ij})$ and $B = (b_{ij})$ $(i,j = 0, 1, 2)$ be the symmetric matrices defining the equations $\sum a_{ij}x_i x_j = 0$ and $\sum b_{ij}v_i v_j = 0$ of a conic-locus S and a conic-envelope E in S_2; and let $a = (a_{00}, a_{01}, a_{11}, a_{02}, a_{12}, a_{22})$ and $b = (b_{00}, b_{01}, b_{11}, b_{02}, b_{12}, b_{22})$ be the points of projective spaces A_5 and B_5 that represent S and E respectively in the usual mappings of the conic-loci and conic-envelopes of S_2 on the points of A_5 and B_5. All the couples (S, E) are thus mapped on the two-way points (a, b) of the space $A_5 \times B_5$.

We now introduce the *basic correspondence* T between conic-loci and conic-envelopes. This is, namely, the irreducible correspondence defined by its generic two-way point (α, β) in $A_5 \times B_5$, where the components α_{ij} of α are independent indeterminates while the components β_{ij} of β are the cofactors of the α_{ij} in the determinant $|\alpha_{ij}|$. The graph of T in $A_5 \times B_5$ is a well defined irreducible two-way algebraic variety $\mathscr{W}$ of dimension five; and we define the *class of complete conics* of S_2 to be that of the couples (S, E) that are represented by points of $\mathscr{W}$, i.e. by all proper specializations (a, b) of (α, β). A set of two-way equations that precisely define T (i.e. $\mathscr{W}$) were first formulated by van der Waerden [**W**, p. 647]. These take the form of the *eight fundamental relations* (defining complete conics) namely

$$\left.\begin{aligned} &\sum_{\alpha=0}^{2} a_{i\alpha} b_{j\alpha} = 0 \quad (i,j = 0, 1, 2;\ i \neq j) \\ &\sum_{\alpha=0}^{2} a_{0\alpha} b_{0\alpha} = \sum_{\alpha=0}^{2} a_{1\alpha} b_{1\alpha} = \sum_{\alpha=0}^{2} a_{2\alpha} b_{2\alpha} \end{aligned}\right\}. \tag{2.1}$$

Van der Waerden formally verified that all the solutions (a, b) of these equations correspond to four projectively distinct types of complete conics (S, E), namely

(i) the *regular* type, with S irreducible and E the envelope of tangents to S,

(ii) the δ-*conic*, for which S is a pair of distinct lines – the *arms* of δ – while E is the *vertex* of this line-pair counted twice,

(iii) the η-*conic*, for which E is a pair of distinct points – the *eyes* of η – while S is the join of these points – the *axis* of η – counted twice; and

(iv) the $\delta\eta$-*conic*, for which S is a repeated line – the *axis* of $\delta\eta$ – while E is a repeated point of this line – the *vertex* of $\delta\eta$.

Notation. In view of the above, we may now introduce the general notation $\widehat{S, E}$ for the complete conic whose locus S and envelope E, with matrices (a_{ij}) and (b_{ij}) respectively, satisfy the eight fundamental relations (2.1).

In what follows we shall often find it convenient to use each of the terms δ-conic and η-conic in the *broad sense*, regarding the δη-conics as specialized members of each of the two families (δ) and (η); but it will always be clear from the context in what sense the terms in question are being used.

2.1. *The standard model* Ω *of complete conics*

We now take the one-way equivalent of the two-way variety $\mathscr{W}$ to be our standard model Ω of the complete conics of S_2; so that Ω is a five-dimensional variety on the Segre product of A_5 and B_5, a V_{10} in a space S_{35}. The generic point of Ω has coordinates $(\alpha_{ij}\beta_{i'j'})$ in S_{35}, where (α, β) is the generic point of $\mathscr{W}$ as previously defined. We may say then that the points of Ω are all those points $(a_{ij}b_{i'j'})$ for which a and b satisfy the eight fundamental relations (2.1).

If the a_{ij} are the coordinates of an arbitrary point of A_5, and if A_{ij} denotes the cofactor of a_{ij} in $|A|$, then we can say that Ω admits the *birational parametric representation*

$$c = (a_{ij}A_{i'j'}) \quad (i,j,i',j' = 0,1,2) \tag{2.1.1}$$

on the space A_5. By virtue of the relations (2.1) which define eight independent primes of S_{35}, we see that Ω lies in a space S_{27}, being the complete intersection of V_{10} by this space. Further, Ω contains fourfolds δ and η, whose points represent δ-conics and η-conics (in the broad sense); and it contains also a threefold δη (common to δ and η) whose points represent the δη-conics.

2.2. *The threefold* δη

Plainly δη is a one:one map of the incident point-lines $\widehat{x,u}$ of S_2. We recall now that the minimum model of these $\widehat{x,u}$ is a variety $V_3^6[7]$, this being a (general) prime section of the Segre product $V_4^6[8]$ of two planes. By use of the parametric equations (2.1.1), it can be verified that

δη *is the transform of* V_3^6 *by all the quadrics of its ambient space* S_7.

Hence δη is of order $6 \times 2^3 = 48$; and it is normal in a space S_{26}.

The properties of the minimum model $V_3^6[7]$ are well known (see, for example, [**S**, §59]). It is simply generated by each of two ∞^2-systems of lines of which any line of the first system represents $\widehat{x,u}$ for which x is fixed while any line of the second system represents $\widehat{x,u}$ for which u is fixed. It follows that δη is simply generated by each of two ∞^2-systems of conics. We shall denote a conic of the first system, representing δη with a fixed vertex, by a, and the plane† of such a conic by α. Similarly we shall denote a conic of the second system, representing δη with fixed axis, by b and the plane containing it by β. It is easy now to verify, and will be clear in any case from our later discussion (§5) of the birational representations of Ω on A_5 and B_5 respectively, that

the fourfold δ *is simply generated by the conic-planes* α *of* δη; *and* η *is simply generated by the conic planes* β *of* δη.

From known results for algebraic equivalence of curves and surfaces on $V_3^6[7]$ given by Severi [**S**, §59] we deduce the following results:

(i) *an equivalence base for curves on* δη *is a pair of conics* (a, b); *and*

(ii) *an equivalence base for surfaces on* δη *is constituted by a pair of surfaces* (ρ, σ), *where* ρ *represents* δη-*conics with vertices on a fixed line of* S_2, *while* σ *represents* δη-*conics with axes through a fixed point.*

† Each conic-plane α of δη maps the δ-conics of S_2 that have a fixed vertex; and, similarly, each conic-plane β of δη maps the η-conics that have a fixed axis.

The surfaces ρ and σ, in fact, are quadric transforms of the two types of cubic scrolls on $V_3^8[7]$, of which the first represents $\widehat{x,u}$ with x on a fixed line and the second represents $\widehat{x,u}$ with u through a fixed point. Plainly ρ is generated by the conics a that meet a fixed b, while σ is generated by the conics b that meet a fixed a. Each of ρ, σ is a surface ${}^2F^{12} \equiv C^4[O^2]$.

Finally, for subsequent use, we introduce a further notation:

The symbols u, v will denote respectively a line that lies in a plane α and a line that lies in a plane β.

A line u represents an involution of δ-conics with a fixed vertex, this involution being degenerate if u touches the conic a in the plane α that contains it; and, similarly, v represents an involution of η-conics with fixed axis.

2.3. *The special van der Waerden parametrization of Ω*

We introduce a result which is basic for all that follows and represents a fundamental simplification of the situation. This is

THEOREM 1. *The varieties Ω, δ, η and δη are all non-singular; and δ and η meet simply, without contact, in δη.*

This result is given both by van der Waerden [**W**, § 1] and by Severi [**S**, § 50], but we shall confine ourselves here to an exposition, with some amplification, of van der Waerden's method of proving it, and to some remarks on further applications of the special parametrization devised by van der Waerden in the course of his proof.

We begin by remarking that the group of self-collineations of S_2 induces a group of self-collineations of Ω; also that this induced group is plainly transitive over each of the four classes of points on Ω that represent projectively distinct types of complete conics. Reflexion then shows that all the results stated in theorem 1 will follow if we can define a suitable set of uniformizing parameters x, y, z, u, v for Ω in the neighbourhood of a chosen point P_0 of δη such that the equations of δ, η and δη in terms of these parameters are $v = 0$, $u = 0$ and $u = v = 0$ respectively.

Suppose then that P_0 is the point of δη representing the δη-conic with equations

$$x_0^2 = 0, \quad u_2^2 = 0,$$

i.e. with coefficient vectors

$$a^0 = (1, 0, 0, 0, 0, 0) \quad \text{and} \quad b^0 = (0, 0, 0, 0, 0, 1).$$

For points of Ω in the neighbourhood of P_0, the coordinates a_{00} and b_{22} will not be zero, so that we may take $a_{00} = 1$ for such points. We may then express the five remaining a_{ij} in terms of five new parameters x, y, z, u, v by the matrix equation

$$(a_{ij}) = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & uv \end{bmatrix} \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x & y \\ x & x^2+u & xy+uz \\ y & xy+uz & y^2+uv+uz^2 \end{bmatrix}. \qquad (2.3.1)$$

Further, the resulting equations expressing the five a_{ij} (other than $a_{00} = 1$) in terms of x, y, z, u, v are rationally reversible, giving

$$\left.\begin{aligned} x = a_{01}, \quad y = a_{02}, \quad z = -A_{12}/A_{22}, \\ u = A_{22}, \quad v = |A|/A_{22}^2 \end{aligned}\right\} \qquad (2.3.2)$$

where A_{ij} denotes, as usual, the cofactor of a_{ij} in the determinant $|A| = |a_{ij}|$.

Equations (2.3.1) and (2.3.2) define a birational correspondence between the points of the space A_5, excluding the prime $a_{00} = 0$, and those of the space – say Σ_5 – of the parameters

x, y, z, u, v; this, in its turn, will transform the birational representation of Ω on A_5, given by equations (2.1.1), into a birational representation of Ω on the space Σ_5. This will have the form

$$c_{ijı'j'} = a_{ij}b_{i'j'} \quad (i,j,i',j' = 0,1,2), \tag{2.3.3}$$

where the a_{ij} are given by (2.3.1) with $a_{00} = 1$, and the $b_{i'j'}$ are the $A_{i'j'}$ with a common factor u removed, explicitly

$$(b_{i'j'}) = \begin{bmatrix} 1 & -x & -(y-xz) \\ 0 & 1 & -z \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} uv & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -x & 1 & 0 \\ -(y-xz) & -z & 1 \end{bmatrix}$$

$$= \begin{bmatrix} uv+vx^2+(y-xz)^2 & -xv+z(y-xz) & -(y-xz) \\ -xv+z(y-xz) & v+z^2 & -z \\ -(y-xz) & -z & 1 \end{bmatrix}. \tag{2.3.4}$$

We now observe that in the representation of Ω on Σ_5 given by (2.3.3), with (2.3.1) and (2.3.4), our chosen point P_0 of $\delta\eta$ is mapped on the origin O of Σ_5; and, further, six of the coordinates $a_{ij}b_{i'j'}$, of the general point of Ω have the values

$$1,\ x,\ y,\ -z,\ u+x^2,\ v+z^2,$$

while all the others can be written as polynomials in these six. It follows from this that x, y, z, u, v are uniformizing parameters for Ω in the neighbourhood of P_0 and that P_0 is therefore a simple point of Ω. Further, a point of Ω in the neighbourhood of P_0 will lie on δ if and only if (a_{ij}) has rank $\leqslant 2$ and $(b_{i'j'})$ has rank 1; and from (2.3.1) and (2.3.4) it follows that this happens if and only if $v = 0$; i.e. the local equation of δ at P_0 is $v = 0$. Similarly the local equation of η at P_0 is $u = 0$; and finally the local equations of $\delta\eta$ near P_0 are $u = v = 0$. This implies that each of δ, η and $\delta\eta$ has a simple point at P_0, also that the tangent [4]'s to δ and η at P_0 are distinct, meeting only in the tangent [3] to $\delta\eta$ at P_0. By what we said earlier, then, theorem 1 has now been completely proved.

The significance of the special parametrization described above is that it provides a birational representation of Ω on an affine space Σ_5, which is based on an arbitrarily chosen point P_0 of $\delta\eta$, that it carries P_0 into the origin O in Σ_5, and that it is regular as between the neighbourhoods of P_0 on Ω and of O in Σ_5. Further it transforms δ, η and $\delta\eta$ into two primes through O and the solid in which these primes meet. Besides the principal use made of it in proving theorem 1, it can be used in suitable cases to reduce problems of computing multiplicities of intersection of varieties on Ω to those – simpler in many cases – of computing the corresponding multiplicities in Σ_5. To facilitate such computations we note a few salient details of the representation, recalling that the chosen point P_0 was taken to represent the $\delta\eta$-conic of S_2 with vertex X_2 and axis X_2X_1.

Let a, b be the two conics of $\delta\eta$ through P_0 and let α, β be the planes of these conics (generators of δ and η respectively); also, in an obvious notation, let OX, OY, OZ, OU, OV be the axes at O. Then

(i) the primes OUXYZ and OVXYZ represent δ and η, while the solid OXYZ represents $\delta\eta$;
(ii) the axis OX represents the conic a, image of $\delta\eta$-conics with X_2 as vertex;
(iii) the axis OZ represents the conic b, image of $\delta\eta$-conics with X_2X_1 as axis;
(iv) the plane OUX represents α, image of δ-conics with X_2 as vertex;
(v) the plane OUZ represents β, image of η-conics with X_2X_1 as axis; and
(vi) the plane OXZ represents a surface σ of $\delta\eta$ (cf. §2.2), image of the $\delta\eta$-conics of S_2 with axes through X_2.

As examples of the computations referred to, we consider two surfaces, both lying on δ, namely the surface τ that images δ-conics with X_2X_1 as one arm and the surface ω that images the δ-conics, with one arm through X_1 and the other through X_2. Each of τ, ω contains the conic b. We find then that each of τ, ω is represented on Σ_5 by a cubic scroll (with directrix in the prime at infinity) which has simple contact with the solid OXYZ along the axis OZ. This means that the conic b counts *twice* as a curve of intersection of each of τ, ω with $\delta\eta$.

3. The two-way model variety $\mathscr{W}$

We now return to the two-way model $\mathscr{W}$, with equations (2.1), which maps all complete conics and we study in particular the sub-varieties of $\mathscr{W}$ that represent the different types of complete conics. To this end we first recall for convenience some basic properties of the well known mappings $S \rightarrow a$ and $E \rightarrow b$ of the conic-loci S and conic-envelopes E of S_2 on the points of the spaces A_5 and B_5 respectively.

First, as already mentioned in §1, the repeated lines of S_2 are mapped on the points of a Veronese surface ϕ of A_5; and, dually, the repeated points of S_2 are mapped on the points of a Veronese surface ψ of B_5. Next, the line-pairs of S_2 are mapped on the points of a cubic primal $M = M_4^3$ of A_5 that passes doubly through ϕ and is simply generated by the conic-planes of ϕ (planes that meet ϕ in conics); and, dually, the point-pairs of S_2 are mapped on the points of a cubic primal $N = N_4^3$ of B_5 that passes doubly through ψ. The basic correspondence T defines a Cremona transformation τ of A_5 into B_5 with equations $b_{ij} = A_{ij}$ $(i,j = 0,1,2)$; and this has inverse τ^{-1} with equations $a_{ij} = B_{ij}$ $(i,j = 0,1,2)$. The quadrics of A_5 through ϕ form a homaloidal system (Φ) which is carried by τ into the primes of B_5; and, conversely, the quadrics of B_5 through ψ form the reverse homaloidal system (Ψ). The transformation τ carries each conic-plane of ϕ into a point of ψ; and similarly τ^{-1} carries the conic-planes of ψ into the points of ϕ.

We now recall that the mappings $S \rightarrow a$ and $E \rightarrow b$ determine a *natural correlation* κ between points of ϕ and ψ. We say, namely, that points P, P′ of these two surfaces correspond in κ if the repeated line L and the repeated point V of S_2 represented by P and P′ are such that V lies on L. To a given point P of ϕ there correspond all the points of a conic on ψ which we denote by $\sigma' = \kappa^{-1}(P)$; and similarly to a point P′ of ψ there correspond all the points of a conic $\sigma = \kappa^{-1}(P')$. The plane π' of σ' maps the point-pairs of S_2 that lie on the line L, and the plane π of σ maps the line-pairs of S_2 that have vertex V. The classes of two-way points (P, P′) of $S_{5,5}$ that represent the different types of complete conics can now be identified as follows:

(i) a *regular* complete conic is mapped in $S_{5,5}$ by a pair (P, P′) such that P does not lie on M, and $P' = \tau(P)$;

(ii) a $\delta\eta$-conic is mapped by a pair (P, P′) such that P lies on ϕ, P′ lies on ψ and P, P′ correspond in κ;

(iii) an η-conic (in the broad sense) is mapped by a pair (P, P′) such that P lies on ϕ and P′ lies in the plane of the conic $\kappa(P)$; and

(iv) a δ-conic (in the broad sense) is mapped by a pair (P, P′) such that P′ lies on ψ and P lies in the plane of the conic $\kappa^{-1}(P')$.

The aggregates of the two-way points defined in (ii), (iii), (iv) constitute the *degeneration sub-varieties* on $\mathscr{W}$.

4. The neighbourhood $\tilde{\phi}$ of ϕ in A_5

Whereas the above mapping of the complete conics of S_2 on the two-way points of $\mathscr{W}$ is one:one without exception, we shall often want to use the birational representations of complete conics on one or other of the spaces A_5 or B_5 (cf. §2.1), which are certainly not unexceptional. As regards the representation on A_5, for example, we note that the regular complete conics of S_2 correspond one:one to the points of the open set $A_5 - M$, and δ-conics (in the strict sense) correspond one:one to the points of the open set $M - \phi$; but about η-conics and $\delta\eta$-conics we can only say that all of them that have a given axis correspond to one and the same point of ϕ. To close this considerable domain of ambiguity we introduce the concept of the *neighbourhood* $\tilde{\phi}$ *of* ϕ *in* A_5; or, more particularly, the concept of *neighbour-points* P_1 *of points* P *of* ϕ.

To this end we envisage ϕ as the base surface of the prime ideal $\mathfrak{p}$ of primals of A_5 that contain it. Let P be a point of ϕ and let α be the tangent plane to ϕ at P. The primals of A_5 that contain ϕ and formally satisfy the condition that they touch at P a given solid Σ through α form a sub-ideal $\mathfrak{p}_1$ of $\mathfrak{p}$. If t is any direction at P that does not lie in α, then t imposes a simple linear condition on primals F^n of $\mathfrak{p}$, of any given order $n \geqslant 2$, to contain it; but all such directions t that lie in a given solid such as Σ impose the same linear condition on the F^n, namely the condition (effectively) that the tangent prime to F^n at P (which already contains α) should contain the solid in question. We may therefore regard the ∞^2 directions at P that lie in Σ but not in α as constituting a single neighbour point P_1 of P, determined by the sub-ideal $\mathfrak{p}_1$. Each point P of ϕ has then ∞^2 such neighbour points, each associated uniquely with a tangent solid Σ to ϕ at P; and these constitute *the neighbourhood* $\tilde{P}_\phi$ *of* P *relative to* ϕ. Finally the whole neighbourhood $\tilde{\phi}$ of ϕ is the aggregate of neighbourhoods $\tilde{P}_\phi$ of points P of ϕ; and it is composed therefore of the ∞^4 points P_1 that constitute these $\tilde{P}_\phi$. Any proper *dilatation* of ϕ – in the accepted meaning of this term – will carry $\tilde{\phi}$ into a four-dimensional algebraic variety on a birational transform of A_5.

Returning now to the Cremona transformation τ of A_5 into B_5, we remark that the neighbour points P_1 of a point P of ϕ are carried by τ into the points P' of the conic plane π' of ψ. Further, for given P, the one:one correspondence between P_1 and P' is such that if P_1 is contained in the quadric nodal cone of M at P, then P' lies on the conic $\kappa(P)$ in π'.

We record then the following:

(i) every η-conic (in the broad sense) is uniquely represented in A_5 by a neighbour point P_1 (relative to ϕ) of a point P of ϕ;

(ii) every $\delta\eta$-conic is uniquely represented in A_5 by a neighbour point P_1 (of a point P of ϕ) that lies on M, i.e. which is such that the solid Σ defining it is a generator of the nodal cone of M at P; and

(iii) the whole neighbourhood $\tilde{\phi}$ of ϕ is a one:one map of η-conics (in the broad sense), and the neighbourhood $\tilde{\phi}_M$ of ϕ on M is a proper one:one map of all $\delta\eta$-conics.

Corresponding results hold for the representation of complete conics on B_5.

5. The representation of Ω on A_5

Returning now to the standard model Ω of complete conics, we prove the following result from which most of our detailed knowledge of geometry on Ω will be derived.

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THEOREM 2. *The variety* Ω *is the projective model of the complete system of cubic primals of* A_5 *that contain the Veronese surface* ϕ.

Proof. From the equations (2.1.1) of the parametric representation of Ω on A_5, we see that the prime sections of Ω are represented on A_5 by a system of cubic primals K, whose equation is

$$\Sigma \lambda_{iji'j'} a_{ij} A_{i'j'} = 0 \quad (i,j,i',j' = 0,1,2), \tag{5.1}$$

where the $\lambda_{iji'j'}$ are arbitrary parameters; and we have already noted (§2.1) that the effective freedom of this linear system is 27 because of the eight fundamental relations (2.1) between the 36 cubic forms $a_{ij} A_{i'j'}$. Since the six forms A_{ij} are a base for quadrics through ϕ, it follows that all the primals K contain ϕ. Further, from the plane representation of ϕ we deduce readily that the freedom of all cubic primals of A_5 that contain ϕ is precisely 27. It follows then that prime sections of Ω are represented in A_5 by the complete system $|K|$ of cubic primals through ϕ. This proves theorem 2. The transformation τ carries $|K|$ into the system $|K'|$ of cubic primals of B_5 through ψ.

The order of Ω will be the grade γ of the complete system $|K| = |L+\Phi|$ where L is a prime of A_5 and Φ is a quadric through ϕ. Hence, since $|\Phi|$ is a homaloidal system, we find that

$$\begin{aligned}\gamma &= (L+\Phi)^5 = L^5+5L^4\Phi+\ldots+\Phi^5 \\ &= 1+5\cdot 2+10\cdot 4+10\cdot 4+5\cdot 2+1 = 102.\end{aligned}$$

Hence: *The order of* Ω *is* 102.

Now consider the degeneration fourfolds δ, η and their intersection $\delta\eta$ on Ω. By virtue of theorem 1 we are able to make the following observation.

The birational transformation $A_5 \to \Omega$ *constitutes a proper dilatation of the surface* ϕ *in* A_5 *into the fourfold* η *on* Ω.

For it carries all the cubic primals of A_5 through ϕ into the prime sections of Ω; it carries each point of A_5 that does not lie on ϕ into a unique point of Ω and no two such points into the same point of Ω; and it carries the ∞^4 neighbour points of ϕ (cf. §4) one:one into the points of the fourfold η on Ω. Further, the primal M, since it passes doubly through ϕ, has to be augmented by the neighbourhood $\tilde{\phi}$ of ϕ to constitute a primal of $|K|$, and it therefore represents the fourfold δ on Ω, which forms with η a complete prime section of Ω. By symmetry, then, each of δ and η must be of order 51 (half that of Ω). Finally, δ and η meet without contact in $\delta\eta$; and $\delta\eta$, as we remarked in §2.2, is of order 48, being a complete quadric transform of the minimum model $V_3^6[7]$ of incident pairs $\widehat{x,u}$ of S_2. Hence:

The fourfolds δ *and* η *on* Ω *are each of order* 51, *and together they constitute a prime section of* Ω; *also* $\delta\eta$ *is a threefold of order* 48 *which spans the common ambient prime of* δ *and* η.

The transformations $A_5 \to \Omega$ and $B_5 \to \Omega$ give ready confirmation of the fact, noted in §2.2, that the conic-planes α and β of $\delta\eta$ generate (simply) δ and η respectively. Thus, for example, the transformation $A_5 \to \Omega$ carries M into δ and each conic-plane of ϕ – a generating plane of M – into a plane of δ; and it carries the conic in which the former plane meets ϕ – as a conic in this plane – into the conic a in which the latter plane meets $\delta\eta$.

6. The conditions μ, ν and their specializations $\bar{\mu}$, $\bar{\nu}$

The complete conics of S_2 that satisfy any given algebraic condition c will be represented on Ω by the points of an algebraic variety – the *condition manifold* of c – which we denote by the same symbol c. In this sense δ, η and $\delta\eta$ are condition manifolds; and we remark that each of δ and η is isolated with respect to linear equivalence on Ω because the primals M and N which represent them in A_5 and B_5 respectively are each uniquely determined by their double surfaces ϕ and ψ.

We now define two types of basic condition on a complete conic $\widehat{S, E}$:

(i) μ is the type of condition that requires the coefficients a_{ij} of S to satisfy a fixed linear condition $\sum k_{ij} a_{ij} = 0$, where the k_{ij} are constants; and

(ii) ν is the type of condition that requires the coefficients b_{ij} of E to satisfy a fixed linear condition.

In geometrical terms any condition μ expresses precisely that the locus S is apolar to a fixed conic-envelope e; while any ν expresses the condition that E is apolar to a fixed conic-locus s. Specializations arise for degenerate e or s. In particular, if e is a repeated point P^2, then μ becomes the condition that S passes through P; and if s is a repeated line L^2, then ν becomes the condition that E contains L. When we wish particularly to refer to conditions of these two special types, we shall distinguish them as follows:

(i′) $\bar{\mu}$ (specialization of μ) is the condition on a complete conic that it pass through a fixed point; and

(ii′) $\bar{\nu}$ (specialization of ν) is the condition on a complete conic that it touch a given line.

Condition manifolds μ and ν on Ω are represented in A_5 by primes and by quadrics Φ respectively; whereas condition manifolds $\bar{\mu}$ and $\bar{\nu}$ are represented in A_5 by *contact primes* of ϕ (each touching ϕ along a conic) and by special members Φ^* of (Φ) which are the *nodal cones of M* at points of ϕ. Each Φ^* is a quadric plane-cone whose vertex is the tangent plane τ to ϕ at a point P; and it is generated by the solids that join τ to the conics on ϕ that pass through P. Further, Φ^* touches M along the cone V_3^3 that projects ϕ from P, being therefore a *contact quadric* of M. Dually, the condition manifolds μ and ν on Ω are represented on B_5 by the quadrics Ψ and by primes respectively, while the $\bar{\mu}$ and $\bar{\nu}$ are represented by the contact quadrics Ψ^* of N and by the contact primes of ψ.

Now let ω denote a prime section of Ω, represented in A_5 and B_5 respectively by a cubic primal K and by a cubic primal K′.

In A_5, with μ now denoting a prime of this space, the basic equivalence relations between K, M and Φ are expressed symbolically in the form

$$K \equiv 3\mu - \tilde{\phi}, \quad M \equiv 3\mu - 2\tilde{\phi}, \quad \Phi \equiv 2\mu - \tilde{\phi} \tag{6.1}$$

and there are analogous relations between K′, N and Ψ in B_5. Since δ and η are represented in A_5 by M and $\tilde{\phi}$ respectively, equations (6.1) lead to the following relations of linear equivalence on Ω, namely

$$\omega \equiv 3\mu - \eta, \quad \delta \equiv 3\mu - 2\eta, \quad \nu \equiv 2\mu - \eta \tag{6.2}$$

and these are equivalent to the formulae

$$\nu \equiv 2\mu - \eta, \quad \mu \equiv 2\nu - \delta \tag{6.3}$$

and

$$\omega \equiv \mu + \nu \equiv \eta + \delta. \tag{6.4}$$

The two formulae (6.3), interpreted numerically, are the *basic characteristic formulae for complete conics*; and it appears from (6.4) that $|\mu+\nu|$ is the complete system of prime sections of Ω.

From the representation of Ω on A_5, we deduce directly the following basic enumerative results

$$\mu^5 = \nu^5 = 1, \quad \mu^4\nu = \mu\nu^4 = 2, \quad \mu^3\nu^2 = \mu^2\nu^3 = 4, \tag{6.5}$$

observing that these generalize the familiar elementary results for $\bar{\mu}^\alpha\bar{\nu}^\beta$ with $\alpha+\beta = 5$.

A useful application of (6.5), combined with the relation (6.4), $\omega \equiv \mu+\nu$, is that these formulae enable us in many cases to compute directly the (virtual) orders of sub-varieties V_d $(0 \leqslant d \leqslant 4)$ of Ω. Suppose, namely, that V_d is known to satisfy a relation of algebraic equivalence on Ω of the form $V_d \equiv F(\mu, \nu)$ where $F(\mu, \nu)$ is a form of degree $5-d$ in μ, ν with integer coefficients. Then, by virtue of (6.4), we may write

$$O(V_d) = (\mu+\nu)^d F(\mu,\nu);$$

and this is evaluated by substituting from (6.5) for each product $\mu^\alpha\nu^\beta (\alpha+\beta = 5)$ in the expansion of the right hand side. Thus, for example, using (6.3), we may write

$$\begin{aligned} O(\delta\eta) &= (\mu+\nu)^3 (2\nu-\mu)(2\mu-\nu) \\ &= -2\mu^5-\mu^4\nu+7\mu^3\nu^2+7\mu^2\nu^3-\mu\nu^4-2\nu^5 \\ &= -2\cdot1-1\cdot2+7\cdot4+7\cdot4-1\cdot2-2\cdot1 \\ &= 48 \end{aligned}$$

in agreement with the result recorded in §2.2.

At this stage, with enumerative results in mind, we must point out that a condition manifold $\bar{\nu}$ on Ω (as distinct from a general ν) touches δ along their intersection, as follows from our previous remark that a quadric Φ^* in A_5 is a contact quadric of M; also, dually, a manifold $\bar{\mu}$ touches η along the threefold they have in common. Thus, for example, whereas the general condition curve $\mu^2\nu^2$ on Ω meets each of δ and η in four distinct points, a condition curve $\bar{\mu}^2\bar{\nu}^2$ has only one intersection – but this counts quadruply – with each of δ and η. As it happens, however, the general system $\bar{\mu}^2\bar{\nu}^2$ is composite, being made up of two similar components, as is clear from the familiar example in the euclidean plane of the circles that touch two given lines. Thus whereas the condition curve of a general system $\mu^2\nu^2$ is an elliptic octavic curve ${}^1C^8$ quadrisecant to each of δ and η, that of a system $\bar{\mu}^2\bar{\nu}^2$ consists of a pair of rational normal quartic curves ${}^0C^4$ which each touch δ (but not each other) at the same point and each touch η at a common point.

6.1. *Some examples of condition manifolds on* Ω

We now note, for future reference, some details about the projective character of the condition manifolds μ, ν, $\bar{\mu}$ and $\bar{\nu}$ on Ω, and of the intersections of these with δ, η and $\delta\eta$. In these examples, as in others to be given later, the results we give are obtained either from the representation of the manifolds in question on A_5 (or B_5) or directly from the systems of conics they represent by use of the parametric equations (2.1.1) of Ω.

(i) *The general* μ. This, being represented in A_5 by a general prime Π, is the projective model of the system of primals cut on Π by primals of $|K|$. It is, therefore, the projective model V_4^{51} of the cubic primals of Π that contain a rational normal curve ${}^0C^4$. The general $\bar{\mu}$, on the other hand, is a $\bar{V}_4^{51}$, projective model of the cubic primals of a contact space $\bar{\Pi}$ that touch ϕ along the contact conic of this space. There are analogous characterizations of ν and $\bar{\nu}$.

(ii) *The general threefold* $\mu\delta$. This maps the line-pairs (δ-conics) of S_2 that are conjugate with respect to a fixed conic-envelope e. It is a V_3^{33}, projective model of the sections of a cubic primal $M_3^3[4]$ – section of M by a prime Π – by cubic primals of Π that pass through the double curve ${}^0C^4$ of M_3^3. Since M_3^3 is the chord locus of ${}^0C^4$, $\mu\delta$ is simply generated by ∞^2 lines of which one lies in each plane α of δ. There are modifications for $\bar{\mu}\delta$ and analogous results for $\nu\eta$ and $\bar{\nu}\eta$.

(iii) *The general* $\mu\eta$. This maps the point-pairs (η-conics) of S_2 whose axes touch a fixed conic-envelope e. It is a V_3^{18} which is simply generated by ∞^1 planes β of η (each representing η-conics with a fixed tangent to e as axis). On the other hand a variety $\bar{\mu}\eta$, representing the point-pairs of S_2 whose axes pass through a fixed point, is a V_3^9 counted twice, which is generated by the ∞^1 planes β of η that meet a fixed conic a of $\delta\eta$. There are analogous results for $\nu\delta$ and $\bar{\nu}\delta$.

(iv) *The general* $\mu\delta\eta$. This is the model of $\delta\eta$-conics whose axes touch a fixed conic-envelope e. It is a surface ${}^5F_2^{24}$, projective model of a curve-system $C^8[A^6, B^2]$; and it is generated by ∞^1 conics b of $\delta\eta$ (represented by lines through A). On the other hand $\bar{\mu}\delta\eta$ is a *surface of contact*, a ${}^2F_2^{12}$ counted twice. This ${}^2F_2^{12}$ is the projective model of a curve-system $C^4[A^2]$; it is generated by the ∞^1 conics b of $\delta\eta$ that meet a fixed conic a; and it is therefore a surface on $\delta\eta$ of the type that we denoted earlier (§2.2) by σ. There are analogous results for $\nu\delta\eta$ and $\bar{\nu}\delta\eta$.

7. Algebraic equivalence bases on Ω, δ, η and $\delta\eta$

In what follows we shall be largely concerned with the classification of systems of algebraic equivalence of varieties of different dimension on each of Ω, δ, η and $\delta\eta$; and, in particular, with the determination of the simplest bases for each such type of equivalence, according to the dimension of the varieties involved. For $\delta\eta$, as indicated in §2.2, the results are already known, a base for curves being a pair of conics (a, b) while that for surfaces is a pair (ρ, σ), as previously defined. Further, the results for either of δ or η will follow by duality from those for the other. Thus we have only to investigate equivalence bases for fourfolds, threefolds, surfaces and curves on Ω, as well as those for threefolds, surfaces and curves on δ (or η).

The varieties Ω and δ are birational transforms of the space A_5 and the (rational) primal M. Since the operations that convert A_5 into Ω and M into the fourfold δ are *proper dilatations*, of the surface ϕ in the first case into the fourfold η, and of the double surface ϕ of M in the latter case into the threefold $\delta\eta$ (cf. §5), we shall allow ourselves to assume the following:

(*a*) for each dimension d $(1 \leqslant d \leqslant 4)$ on Ω, the relation of equivalence of d-folds on Ω has a finite basis, and the same holds for the equivalence of k-folds on δ for $k = 1, 2, 3$; and

(*b*) on Ω, as also on δ, the base numbers for varieties of complementary dimension, say $d + d' = 5$ or $k + k' = 4$, are equal [cf. S, §6].

This means that, if we can find bases for fourfolds and threefolds on Ω, then we can select dual bases for curves and surfaces on Ω, subject only to the condition that the intersection matrix for each proposed pair of dual bases is non-singular.

For fourfolds on Ω, a direct proof that the pair (μ, ν) constitutes such a base is easy and we shall give this in the following section. For threefolds on Ω and for threefolds and surfaces on δ, however, we need to apply the rather sophisticated machinery of the so-called *method of degenerate collineations*; and it will be convenient to postpone the detailed theory and application of this method to a later stage (§11), where the problems to be solved can be treated together.

We propose, therefore, pending this later discussion, to put forward as *prospective* bases for threefolds on Ω and for threefolds and surfaces on δ, the bases that naturally suggest themselves when the properties of the two dilatations concerned are taken into account. On occasion, application of these prospective bases will be made in anticipation of their subsequent formal verification.

7.1. *Equivalence bases for fourfolds and curves on* Ω

Let U be any irreducible fourfold other than δ or η on Ω. Then U will be represented on A_5 by an irreducible primal $\bar{U}$. Let $n \geqslant 1$ be the order of $\bar{U}$ and let $i \geqslant 0$ be the multiplicity of ϕ on $\bar{U}$. Then plainly $2i \leqslant n$; for, in the contrary case, $\bar{U}$ would contain every chord of ϕ; and hence, being irreducible, it could only be the primal M, image of δ, contrary to hypothesis. Now $\bar{U}$ is a member of the complete linear system $|\bar{U}|$ of primals of A_5 that are of order n and have ϕ as i-tuple surface; and hence U is a member of the corresponding complete linear system $|U|$ on Ω. Further $|\bar{U}|$ contains members which each consist of $n-2i$ primes of A_5 together with i quadrics Φ through ϕ; whence $|U|$ contains members which each consist of $n-2i$ manifolds μ and i manifolds ν .This means that every irreducible fourfold on Ω, other than δ or η, is linearly equivalent to a composite fourfold of the form $p\mu+q\nu$, where p and q are non-negative integers. As regards δ and η, moreover, we have the virtual equivalence relations

$$\delta \sim 2\nu-\mu \quad \text{and} \quad \eta \sim 2\mu-\nu$$

(cf. (6.3)). By combining the above results we have

THEOREM 3. *A pair of condition manifolds* μ *and* ν *constitutes a complete base for the algebraic equivalence of fourfolds on* Ω.

If $c \sim p\mu+q\nu$, where p and q are integers, then p, q are *characteristics* of the condition c with respect to the equivalence base (μ, ν). To find p, q for a simple condition we first select a convenient base for curves on Ω, a base complementary to the base (μ, ν) for fourfolds. This will consist of two members (cf. §7); and subject to certain observations we make below, we can take these to be (cf. §2.2)

(i) a line u lying in a generating plane of δ, representing an involution of line-pairs of S_2 with a fixed vertex, and

(ii) a line v lying in a generating plane of η, representing an involution of point-pairs of S_2 with a fixed axis. We find then that the two bases (μ, ν) and (u, v) have the non-singular intersection matrix

$$\begin{bmatrix} \mu u & \mu v \\ \nu u & \nu v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{7.1.1}$$

Also, if $c \sim p\mu+q\nu$ as above, then

$$cu = (p\mu+q\nu)\,u = p$$

and

$$cv = (p\mu+q\nu)\,v = q. \tag{7.1.2}$$

Hence the characteristics p, q *of the condition* c *are the numbers of members the system* (c) *has in common with a system* (u) *of* δ*-conics and with a system* (v) *of* η*-conics respectively.*

When the characteristics p_i, q_i have been found for each of five simple conditions c_i $(i = 1, \ldots, 5)$, and provided always that these c_i are simultaneously satisfied by only a finite number of complete conics (cf. Appendix B on *Halphen conditions*), the (virtual) number m of complete conics that satisfy the c_i is given by

$$m = \prod_{i=1}^{5} (p_i\mu+q_i\nu), \tag{7.1.3}$$

where the product on the right is to be evaluated by use of the values of $\mu^{\alpha}\nu^{\beta}$ $(\alpha+\beta = 5)$ as given in (6.5). Thus, for example, if c is the condition that a conic should touch a given (complete) conic t, then we find that $cu = cv = 2$; so that

$$c \sim 2\mu+2\nu \tag{7.1.4}$$

and hence $$c^5 = (2\mu+2\nu)^5 = 3264.$$

The formula $c \sim 2\mu+2\nu$, which is the basis for this classical enumerative result, was originally derived by allowing the assigned conic to degenerate into a δ-conic or an η-conic.

7.2. *Remarks on the base for curves*

We now give some brief and informal remarks about our choice of an equivalence base for curves on Ω. Consider then a continuous system $(C(\lambda))$ of curves in A_5, such that, for a general value of the parameter λ the curve $C(\lambda)$ does not meet the surface ϕ, while a particular curve $C(\lambda_0)$ of the system has a simple intersection P_0 with ϕ. Further, let $(D(\lambda))$ denote the continuous system of curves on Ω that corresponds to $C(\lambda)$. As $\lambda \to \lambda_0$, the curve $D(\lambda)$ tends to a composite curve of which one part is the proper transform, say D, of $C(\lambda_0)$ – of order one less than that of $D(\lambda)$ – while the other is a line v in the plane β which represents the neighbourhood of P_0. This line v must represent, as we know, the section of the neighbourhood of P_0 by a certain [4], say H_4, through the tangent plane τ_0 to ϕ at P_0; and it may be shown, in fact, that H_4 is determined (in general) by the condition that it must contain not only τ_0 but also the tangent plane at P_0 to the surface generated by the continuous system $C(\lambda)$. By an obvious generalization, we see that *a curve* C *in* A_5 *that meets* ϕ *simply at each of k points gives rise on* Ω *to a total transform which consists of the proper transform* D *of* C *together with k lines v.*

We note that a general line of A_5 – representing a general pencil of conics of S_2 – corresponds on Ω to a twisted cubic curve p (its transform by $|K|$); and a chord of ϕ, representing in A_5 an involution of line-pairs of S_2 with fixed vertex, is represented on Ω by a line u. By what we have said above, it follows that we have on Ω the equivalence relation

$$p \equiv u+2v. \tag{7.2.1}$$

By duality, then, if q is the twisted cubic on Ω that represents the conics of a general range in S_2, we also have

$$q \equiv 2u+v. \tag{7.2.2}$$

Thus (p, q) could also be taken as a basis for the equivalence of curves on Ω; and this is in fact the basis chosen by Severi [S, §51].

In our choice of (u, v), in preference to (p, q), we do not overlook the apparent anomaly that the lines u are all confined to the fourfold δ, while the lines v are all contained in η. This requires, in particular, that virtual values have to be computed for the numerical symbols $u\delta$ and $v\eta$. However, by noting such well known properties of pencils and ranges of conics as

$$p\delta = 3, \quad p\eta = 0, \quad q\delta = 0 \quad \text{and} \quad q\eta = 3,$$

and by using the equivalence relations (7.2.1) and (7.2.2), we find at once that

$$u\delta = v\eta = -1. \tag{7.2.3}$$

7.3 *Equivalence bases for threefolds and surfaces on* Ω

In looking for possible equivalence bases for threefolds on Ω, an obvious suggestion is that, since (μ, ν) is a base for fourfolds, then a base for threefolds might consist of the triad μ^2, μν, ν^2, these three varieties being represented in A_5 respectively by a general solid of A_5, a quadric threefold V_3^2, section of a quadric Φ (through ϕ) by a prime of A_5, and a quartic threefold V_3^4, the intersection of a pair of quadrics through ϕ. As it happens, however, a threefold μν is equivalent on Ω to two members of a simpler equivalence system on Ω, a system that we shall denote by $\widehat{\mu\nu}$, of which any member is represented in A_5 by a solid T that meets ϕ in a conic k and (as it normally does) in one residual point P; in other words, we shall have the equivalence relation

$$\mu\nu \equiv 2\widehat{\mu\nu}. \tag{7.3.1}$$

It is a simple matter now to verify the following result:

A variety $\widehat{\mu\nu}$ *on* Ω *represents the system of conics of* S_2 *that have a given line* q *as polar of a given point* Q.

This is a self-dual system of conics, in agreement with the fact that the Cremona transformation τ carries the solid T into a solid T′ of B_5 that meets ψ in a conic k' and a residual point P′. In the interpretation of $\widehat{\mu\nu}$, as given above, the repeated line whose image is P is the line q of S_2, while the point Q is the vertex of the pencil of repeated lines that are mapped by the points of k.

To verify (7.3.1) we note that the V_3^2 which represents a variety μν on A_5 meets ϕ in a prime section $^0C^4$ of this surface; and this $^0C^4$ can degenerate into a pair of conics k_1, k_2 of ϕ which meet in a single point P; and the pair of solids T_1 and T_2 that join P to k_1 and k_2 respectively constitutes a proper specialization of V_3^2. This proves (7.3.1).

In advance of the formal analytical proof due to van der Waerden, to be given later in §11.2, we now announce the following result:

A triad of varieties (μ^2, $\widehat{\mu\nu}$, ν^2) *constitutes a complete equivalence base for threefolds on* Ω.

To appreciate the geometrical significance of this result, it is useful to indicate briefly the alternative way in which it can be approached, the method pursued (in a generalized form) by Severi [**S**, §§39–41]. If V is a threefold on Ω that is represented in A_5 by a threefold $\overline{V}$, then we may suppose that the equivalence class of V on Ω depends essentially on the behaviour of $\overline{V}$ relative to the surface ϕ that is being dilated into the fourfold η of Ω. We are supposing, in other words, that the 'total' transform of $\overline{V}$ on Ω will consist in part of its 'proper' transform V together with one or more threefolds, lying entirely on η, which correspond only to 'neighbourhoods', not explicit threefolds, of varieties common to $\overline{V}$ and ϕ. We recognize, in particular, three main types of behaviour of $\overline{V}$ with respect to ϕ, namely (*a*) $\overline{V}$ may meet ϕ only normally, i.e. in a finite set of points, or (*b*) $\overline{V}$ may meet ϕ in a curve, together possibly with some isolated points, or (*c*) $\overline{V}$ may contain ϕ entirely, possibly to a certain multiplicity. As the simplest cases of these three types of behaviour, we consider those in which $\overline{V}$ is respectively

(i) a solid Σ of A_5 which meets ϕ in isolated points only,

(ii) a solid T of A_5 which meets ϕ in a conic k and (as normally) in one further isolated point, and

(iii) a quartic threefold which is the intersection V_3^4 of two quadrics Φ through ϕ.

These three represent threefolds on Ω of the types we have denoted by μ^2, $\widehat{\mu\nu}$ and ν^2 respectively.

The possibility that these three may constitute a base for threefolds on Ω rests now on certain plausible assumptions. It may be assumed, for example, that a threefold $\bar{V}$ of order n which meets ϕ in a curve which is equivalent on ϕ to s of the conics k, represents a threefold V on Ω which is equivalent to s of the varieties $\widehat{\mu\nu}$ together with $n-s$ varieties μ^2; or again, that a $\bar{V}$ of order n which passes t-tuply through ϕ will represent a threefold V on Ω which is equivalent to a combination of t of the varieties ν^2 together with $n-4t$ varieties μ^2. Under these tentative assumptions, the equivalence classes of threefolds on Ω which lie entirely on η would be expressible as differences of threefolds on Ω that are properly represented by (effective) threefolds in A_5. These remarks are perhaps enough to indicate the alternative approach to our result (to be formally proved in §11.2 by the method of degenerate collineations) that μ^2, $\widehat{\mu\nu}$ and ν^2 are a complete base for threefolds on Ω.

Before passing on from the above discussion of threefolds on Ω, we have one further remark to add about an important specialization of the condition manifold $\widehat{\mu\nu}$. A solid T of A_5 which passes through a conic k of ϕ may be such that its residual intersection with ϕ comes to lie on k itself; in which case T touches ϕ at a point P of k. In this case, the condition manifold it represents on Ω will be a specialization of $\widehat{\mu\nu}$ which we shall denote by $\overline{\mu\nu}$. We verify easily that:

The specialization $\overline{\mu\nu}$ of $\widehat{\mu\nu}$ maps the system of conics of S_2 *that touch a given line at a given point of this line.*

The reader should be warned, however, that in some of the earlier literature the symbol $\widehat{\mu\nu}$ has been used to denote the specialization that we denote here by $\overline{\mu\nu}$.

We now consider possible bases for the algebraic equivalence of surfaces on Ω. From (a) and (b) of §7, we can assert first that finite bases of the kind in question do in fact exist, and secondly that the relevant base number is three, equal to that for threefolds (varieties of the complementary dimension on Ω). As two members of a prospective base triad for surfaces, we naturally choose a generating plane α of δ and a generating plane β of η respectively. For the third member we choose a surface, to be denoted by χ, which represents on Ω the (self-dual) system of conics of S_2 that have a common self-polar triangle. The surface χ is represented in A_5 by a trisecant plane of ϕ (and in B_5 by a trisecant plane of ψ); it is a Del Pezzo surface, projective model of a plane curve-system $C^3[O_1, O_2, O_3]$; it possesses a closed hexagon of lines of which one triad of alternate sides are lines u – the intersection of χ with δ – while the other triad consists of lines v – the intersection of χ with η; and the six vertices constitute the intersection of χ with $\delta\eta$.

If g is the model surface on Ω of a general linear net of conics of S_2 – representing a general plane π of A_5 – then, as π is specialized to a trisecant plane ABC of ϕ, g becomes specialized into a surface χ together with the three planes β that represent (on η) the neighbourhoods of A, B, C in A_5. Thus

$$g \equiv \chi + 3\beta \tag{7.3.2}$$

and dually, if h is the model in Ω of a general linear net of conic-envelopes of S_2, then

$$h \equiv \chi + 3\alpha. \tag{7.3.3}$$

Finally, we now verify easily that the two triads $(\mu^2, \widehat{\mu\nu}, \nu^2)$ and (α, χ, β) have the non-singular intersection matrix

$$\begin{bmatrix} \mu^2\alpha & \mu^2\chi & \mu^2\beta \\ \widehat{\mu\nu}\alpha & \widehat{\mu\nu}\chi & \widehat{\mu\nu}\beta \\ \nu^2\alpha & \nu^2\chi & \nu^2\beta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7.3.4}$$

and this gives the following result:

The triad (α, χ, β) is a complete equivalence base for surfaces on Ω.

7.4. *Equivalence bases for threefolds, surfaces and curves on δ and on η*

It now remains for us to select prospective equivalence bases for threefolds, surfaces and curves on δ and on η, remembering that (by S_2 duality) bases for δ will determine corresponding bases for η. Consider then δ which is the image manifold of the geometric variable consisting of an unordered pair of lines (p, q) of S_2 for which the vertex is required to be determinate when $p = q$. In A_5, δ is represented by the primal M, each conic-plane π of φ representing a generating plane α of δ, image of line-pairs of S_2 with fixed vertex; and in each such plane α there lies a conic a of δη, which corresponds to the neighbourhood on M of a point of φ. Apart from δη itself, which is an isolated threefold on δ, the two most obvious types of threefold on δ are as follows:

(i) a threefold J which represents the δ-conics of S_2 with vertex on a fixed line, and
(ii) a threefold K which represents the δ-conics that pass through a fixed point.

Plainly $K = \bar{\mu}\delta$ is a specialization of μδ.

We choose the pair (J, K) *as our prospective base for threefolds on* δ; and, later (§11.3), it will be proved that (J, K) is in fact a complete base.

As a complementary equivalence base for curves on δ, we take a line u and a conic b (since δ contains no lines v), and we verify then that the couples (J, K) and (u, b) have the non-singular intersection matrix

$$\begin{bmatrix} Ju & Jb \\ Ku & Kb \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \qquad (7.4.1)$$

Thus:

The couple (u, b) *is an equivalence base for curves on* δ.

We now add some remarks about the threefolds J and K on δ, of which the former will be of special importance to us in the sequel.

The threefold J is the model on δ of the line-pairs of S_2 that have vertices on a fixed line p, and it is therefore a rational scrollar variety generated by the ∞^1 planes α each of which maps the δ-conics of S_2 with vertex at a fixed point of p. These planes α, in fact, are those which meet a fixed conic b (representing the δη-conics with p as axis); and the conics a which lie in these planes α generate a surface ρ (cf. §2.2), the intersection of J with δη. Further, J possesses a simple directrix surface ${}^0R^5$ – to be denoted by τ – which maps the δ-conics that have p as one arm; and it also possesses a family of simple directrix curves ${}^0C^4$, lying on ρ but not meeting τ, each of which represents the δη-conics, with vertex on p, that pass through a fixed point of S_2. It follows easily from this that:

The threefold J *is a rational scrollar variety* ${}^0R^9_3$ *of the most general type, with an* [11] *as its ambient space.*

This can be verified directly from the parametric equations of J. J is represented in A_5 by the cone ${}^0R^3_3$ that projects φ from a point P of itself, this cone being generated by the planes of conics on φ that pass through P and having the tangent plane of φ at P as its directrix plane. (This plane corresponds in A_5 to the directrix surface τ of J.) The special significance of J is that it is a base for planar threefolds on δ (loci of ∞^1 planes α); for if C is any curve of order n in S_2, then plainly the model of the δ-conics of S_2 that have their vertices on C is algebraically equivalent on δ to nJ. In particular, we have on δ the equivalence relation

$$\nu\delta \equiv 2J, \qquad (7.4.2)$$

for $\nu\delta$ represents in S_2 the δ-conics that are apolar – as envelopes – to a conic locus s; and these are the δ-conics with vertices on s. On M the threefold $\nu\delta$ is represented by the variety generated by conic-planes of ϕ that touch a prime section ${}^0C^4$ of ϕ.

To exhibit $\delta\eta$ as a threefold which is algebraically dependent on J and K on δ, it is tempting to write, with the use of (6.3) and (7.4.2),

$$\delta\eta \sim (2\mu-\nu)\delta \sim (2K-2J); \tag{7.4.3}$$

but strictly this is an equivalence relation on Ω (rather than on δ). However, since the equivalence systems $\mu\delta$, $\nu\delta$ and $\eta\delta$ are all effective on δ, and since the equivalence $\nu+\eta \equiv 2\mu$ on Ω is valid in the narrowest sense – the existence of continuous systems connecting members of both sides – we infer that (7.4.3) is also valid as an equivalence relation on δ.

We now come to the business of selecting a *prospective equivalence base for surfaces on* δ.

Our choice for such a base, later to be confirmed in §11.3, is the triad of surfaces†

$$(\alpha, \tau, \omega)$$

which we define as follows:

α is a generating plane of δ, model of δ-conics with a fixed vertex;

τ is a rational quartic scroll ${}^0R^5$, model of δ-conics with one fixed arm; this is represented in A_5 by a tangent plane τ^* of ϕ; it possesses a directrix conic b; and its generators are lines u, each representing δ-conics with a fixed vertex on the fixed arm; and

ω, which is a surface ${}^3F^{14}$, projective model of a curve-system $C^4[O_1, O_2]$, which maps δ-conics with one arm passing through a fixed point A, and the other passing through a fixed point B. The surface ω is represented on M by a quadric surface ω^* which touches ϕ at a fixed point P (being the residual section of M by a solid Σ through the tangent plane to ϕ at P); and it is simply generated by each of two pencils of cubic curves – represented on ω^* by its two systems of generators – each of the cubic curves representing the δ-conics of the system for which one of the arms is fixed.

We find, then, that the intersection matrix of the triad (α, τ, ω) with itself is given by

$$\begin{bmatrix} \alpha\alpha & \alpha\tau & \alpha\omega \\ \alpha\tau & \tau\tau & \tau\omega \\ \alpha\omega & \tau\omega & \omega\omega \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \tag{7.4.4}$$

which is of determinant -1. Thus:

The triad (α, τ, ω) *is an equivalence base for surfaces on* δ.

Finally, to carry over from equivalence bases on δ to those on η, we only need to adopt a suitable notation convention. This will be, namely, that if any type of variety on δ has been denoted by a certain symbol, as for example J, K, τ or ω, then the analogous (dual) type of variety on η (if not already named) will be denoted by the same symbol with a prime, as, for example, J′, K′, τ' or ω'. Thus the *bases for threefolds, surfaces and curves on* η will be (J′, K′), (β, τ', ω') and (v, a) respectively.

† For reasons given in our footnote to §11.4, we reject Severi's assertion [S, pp. 317–319] that a minimum base for surfaces on δ is composed of four elements.

7.5. *Some special types of surface on* δ *and on* η

It is convenient at this stage to list some of the more significant types of surface on δ or on η and to exhibit their algebraic dependence on the respective bases (α, τ, ω) and (β, τ', ω') for surfaces on δ and on η. As usual, we need only describe those on δ, their analogues on η being inferred by duality.

Consider then δ, represented in A_5 by M, with its sub-variety $\delta\eta$ represented by the neighbourhood of ϕ on M. We make the following preliminary observation.

In the representation of δ *on* M, *the neighbourhood on* M *of a conic k of* ϕ *corresponds to a surface* σ *on* $\delta\eta$ (cf. §2.2), *each point of k giving rise to a generating conic b of* σ.

There is an analogous result for a surface ρ on $\delta\eta$ in terms of the representation of η on N in the space B_5.

Further, as regards σ, we observe that a surface $\mu\delta\eta$ is the model of the $\delta\eta$-conics of S_2 that are apolar to a fixed conic envelope *e* in S_2, such therefore that their axes touch *e*; whence, by allowing *e* to degenerate into a point-pair, we obtain the equivalence relation

$$\mu\delta\eta \equiv 2\sigma \tag{7.5.1}$$

valid on $\delta\eta$. Dually we have

$$\nu\delta\eta \equiv 2\rho. \tag{7.5.2}$$

A different equivalence classification for the surface σ on δ can be derived as follows. We recall first that a surface ω, model of δ-conics of S_2 whose arms pass through fixed points A, B respectively, is represented on M by a quadric surface ω^*, residual section of M by a solid Σ through a tangent plane τ^* of ϕ. If Σ is made to vary about τ^* so as ultimately to contain a conic-plane π of ϕ, then ω^* will degenerate into π counted twice together with the neighbourhood on M of the conic *k* in which π meets ϕ. By our first observation above, it follows then that ω degenerates into a plane α counted twice together with a surface σ. Hence

$$\omega \equiv 2\alpha + \sigma \tag{7.5.3}$$

on δ; and the result is compatible with the orders 1, 12 and 14 of α, σ and ω.

In a later section (§10) we shall have something to say about such obvious surface types as $\mu^2\delta$, $\widehat{\mu\nu}\delta$ and $\nu^2\delta$; but here we restrict ourselves to the following types on δ.

(i) *The surface* ζ (*of an* ∞^5-*system*): *a* ${}^3F^{16} \equiv C^4[—]$. This maps the (unordered) pairs of tangents to a conic envelope *e* of S_2. It is represented on M by a Veronese surface ζ^* that meets ϕ in a prime section ${}^0C^4$; and ζ^* is the locus of the points of intersection of pairs of tangent planes to ϕ at points of ${}^0C^4$. (Two tangent planes to ϕ meet always on M.) By making *e* degenerate into a point-pair we deduce the equivalence relation

$$\zeta = 2\alpha + \omega. \tag{7.5.4}$$

(ii) *The surface* κ (*of an* ∞^4-*system*): *a scroll* ${}^0R^6 \equiv C^4[O^3, O_1]$ *where* O_1 *is a neighbour point of* O. This maps the line-pairs of S_2 that correspond in a fixed harmonic homology; and it is mapped on M by the quadric cone K* that joins a conic *k* of ϕ to a point P of ϕ (not on *k*). By making the vertex of the homology tend to a point on the axis (supposed to be kept fixed), we obtain the equivalence relation

$$\kappa \equiv \alpha + \tau. \tag{7.5.5}$$

(iii) *The surface* κ_1 (*of an* ∞^5*-system*): *a* ${}^2F^{12} \equiv C^4[O^2]$. *This maps the pairs of lines of* S_2 *that correspond in a general* (*not harmonic*) *homology*. It is represented on M by a Veronese surface κ_1^* which touches ϕ along a conic k and also at a point P not on k. We find that

$$\kappa_1 \equiv 2\kappa \equiv 2\alpha + 2\tau. \tag{7.5.6}$$

(iv) *The surface* κ_2 (*of an* ∞^8*-system*): *a* ${}^7F^{24} \equiv C^6[O_1^2, O_2^2, O_3^2]$. This maps the pairs of lines of S_2 that correspond in a general collineation of S_2. It is represented on M by a Veronese surface κ_2^* that touches ϕ at each of three points. Here we find, by a degeneration of the collineation, that

$$\kappa_2 \equiv \omega + 2\tau. \tag{7.5.7}$$

In stating the above results we have implied the existence and properties of certain types of surfaces on M, in particular the two types of Veronese surfaces κ_1^* and κ_2^*. In this connection we observe that M can be regarded as a (2, 1) projection – from a suitably chosen plane ε – of the standard model $V_4^6[8]$ of all the *ordered* line-pairs (or point-pairs) of S_2. The ordered pairs of lines of S_2 that correspond in any collineation of S_2 are mapped on V_4^6 by the points of a Veronese surface (of an ∞^8-family); and in particular the coincident line-pairs of S_2 are mapped by the points of a particular Veronese surface Δ, the *diagonal surface* of the representation. The existence and properties of surfaces on M can then be derived from those of surfaces on V_4^6 by projection from ε, taking into account any special relations that the latter may have to the diagonal surface Δ and to the plane of projection ε. The stated properties of κ^*, κ_1^* and κ_2^* have all been confirmed in this way.

8. Simply infinite systems of complete conics

If $\Sigma(m,n)$ is any irreducible system of complete conics which is of indices m, n (the numbers of members of the system that pass through a given point or touch a given line), then we may suppose, in view of the results given in §7.1, that the image curve of $\Sigma(m,n)$ on Ω belongs to the equivalence system $mu+nv$, where m, n are integers and (u,v) is our base for curves on Ω. We shall denote this image curve by $C(m,n)$. Since u and v are lines, then $C(m,n)$ is of order $m+n$; and by (7.1.7) it meets δ and η in sets of $2n-m$ and $2m-n$ points respectively, provided only that these numbers are non-negative so that $C(m,n)$ does not lie on δ or η.

$C(m,n)$ is represented on A_5 by a curve C^m, of order m, which meets ϕ in $2m-n$ points; and in B_5 by a curve C^n which meets ψ in $2n-m$ points. The lines $u = C(1,0)$ and $v = C(0,1)$ are represented respectively in A_5 and in B_5 by a chord of ϕ and by a chord of ψ.

For $(m,n) = (1,2)$ and $(m,n) = (2,1)$ we get the two families of twisted cubic curves on Ω that map *ordinary pencils and ordinary ranges* of conics in S_2. Curves of the first family are represented in A_5 by lines not meeting ϕ and in B_5 by conics trisecant to ψ; while those of the second family have analogous representations in B_5 and in A_5.

For $(m,n) = (1,1)$, we get the ∞^6-family of conics on Ω, unisecant to each of δ and η, that map the *pencil-ranges* (double-contact pencils) of S_2. These conics are represented in A_5 by the lines unisecant to ϕ; but among them are ∞^5 special members, images of the *four-point contact pencil-ranges* of S_2, which each meet $\delta\eta$ in one point and are represented in A_5 by the lines which each meet ϕ in a point and lie on the tangent cone to M at that point.

Some special interest attaches to self-dual families of complete conics, those for which $m = n$. Each of these is mapped on Ω by a curve of order $2m$, m-secant to each of δ and η; and it corresponds in A_5 to a C^m which is m-secant to ϕ, and in B_5 to another C^m which is m-secant to ψ. The simplest example of a self-dual family is the pencil-range $\Sigma(1, 1)$. The next simplest self-dual family is the quadratic system $\Sigma(2, 2)$, now to be considered.

The quadratic self-dual family $\Sigma(2, 2)$. This is the system that is represented in A_5 by a general conic bisecant to ϕ, and on Ω by a curve ${}^0C^4$ that meets each of δ and η in two points. We find, then, that the conics of $\Sigma(2, 2)$ all have double-contact with each of two fixed conics s, s'; and they form, in fact, one of the three irreducible systems of conics having double contact with s and s', each of these three systems being associated with one vertex of the common self-polar triangle of s and s'. Thus, in general, the equation of a system $\Sigma(2, 2)$ can be taken in the form

$$ax^2+by^2+cz^2+(my+nz)^2 = 0, \quad \text{with} \quad \frac{m^2}{b-a}+\frac{n^2}{c-a}+1 = 0. \tag{8.1}$$

If (8.1) is written in the alternative form

$$x^2+y^2+z^2+(\mu y+\nu z)^2 = 0 \quad \text{with} \quad \frac{\mu^2}{b-a}+\frac{\nu^2}{c-a}-\frac{1}{a} = 0,$$

we find that

$$\frac{\mu}{\nu} = -\frac{b-a}{c-a}\frac{n}{m},$$

which shows that the chords of contact of a conic of $\Sigma(2, 2)$ with the two fixed conics s, s' correspond in an involution of lines through the reference point X. In terms of non-Euclidean geometry with one of s, s' as absolute conic, the three systems $\Sigma(2, 2)$ defined by s and s' are the systems of *focal circles* of the other conic of the pair.

It may be noted incidentally that the conics of S_2 that have double contact with a fixed conic s are mapped in A_5 by points of the cone projecting ϕ from the image point of s. Thus any two such cones, with distinct vertices, will meet in a triad of conics, bisecant to ϕ, each representing a system $\Sigma(2, 2)$.

9. Doubly infinite systems of complete conics

An ∞^2-system of conics may mean, *a priori*, either a system of conic-loci (or envelopes) defined by a given surface in the space A_5 (or B_5), or a system of complete conics defined by a given surface on Ω; but only in the latter case does it come directly within the scope of our enumerative geometry based on Ω. Supposing, however, that a system Σ of the former kind is defined, for example, by an irreducible surface F_1 in A_5 then we can replace it, for our purposes, by the *associated system* Σ of complete conics, where Σ is the system defined by the proper transform F on Ω of F_1 in A_5, provided only that F_1 is not the surface ϕ in A_5 that maps the repeated lines of S_2. More particularly, F is the projective model, birationally equivalent to F_1, of the sections of F_1 by primals of the system $|K|$ that represents the system of prime sections of Ω.

We now recall (cf. §7.3) that a base for surfaces on Ω consists of a pair (α, β) of planes representing respectively δ-conics with fixed vertex and η-conics with fixed axis, together with a surface χ – a sextic Del Pezzo surface – representing a (self-dual) family of complete conics that have a fixed self-polar triangle. This means that every surface on Ω is algebraically equivalent to a combination $p\alpha+q\chi+r\beta$, where p, q, r are non-negative integers. In what follows we propose to record equivalence valuations of this kind for the image surfaces on Ω of a selection of important ∞^2-systems of conics; and we begin with the basic surfaces μ^3, $\mu^2\nu$, $\mu\nu^2$ and ν^3 on Ω.

A surface μ^3 on Ω, being represented in A_5 by a plane not meeting ϕ, arises from a general linear net of conics (conic-loci) of S_2; and, dually, a surface ν^3 arises from a general linear net of conic-envelopes of S_2. By (7.3.2) and (7.3.3) we have the equivalence relations

$$\mu^3 \equiv \chi + 3\beta \quad \text{and} \quad \nu^3 \equiv \chi + 3\alpha. \tag{9.1}$$

On the other hand, a general surface $\mu^2\nu$ on Ω is represented on A_5 by a quadric surface which meets ϕ in four points; and it is therefore equivalent on Ω to $2\mu^3 - 4\beta$ which reduces by (9.1) to $2\chi + 2\beta$. Whence, and by duality, we may write

$$\mu^2\nu \equiv 2\chi + 2\beta \quad \text{and} \quad \mu\nu^2 \equiv 2\chi + 2\alpha. \tag{9.2}$$

Also, from its representation in A_5, we find easily that $\mu^2\nu$ is a surface ${}^4F_2^{14}[11]$, projective model of a plane curve system $C^5[O_1^2, O_2^2, O_3, O_4, O_5]$; and there is a similar result for $\mu\nu^2$.

From (9.1) and (9.2) we deduce that μ^3, $\mu^2\nu$, $\mu\nu^2$ and ν^3 are connected by the virtual equivalence relation

$$2\mu^3 - 3\mu^2\nu + 3\mu\nu^2 - 2\nu^3 = (\mu - \nu)(2\mu^2 - \mu\nu + 2\nu^2) \sim 0. \tag{9.3}$$

We now observe that relations of equivalence on δ, and in particular those relating to surfaces on δ such as we found in § 7.5, will be valid also on Ω; but equivalence systems on δ will generally be contained in more ample equivalence systems on Ω. Thus, for example, a surface ω on δ, represented in A_5 by a quadric surface ω^* which lies on M and touches ϕ, will be equivalent on Ω to the surface that is represented in A_5 by a general quadric surface that touches ϕ. Further, since this latter surface can degenerate into a tangent plane τ^* together with a plane not meeting ϕ, we deduce (using (9.1)) that, on Ω,

$$\omega \equiv \tau + (\chi + 3\beta). \tag{9.4}$$

We next look to find the equivalence valuation on Ω of a surface τ (a rational quintic scroll) which corresponds to a tangent plane τ^* of ϕ in A_5. The plane τ^* has four coincident intersections with ϕ at its point of contact P, and it is therefore a specialization in A_5 of a trisecant plane of ϕ which has come to have four coincident intersections with ϕ at P. The surface χ on Ω which corresponds to the trisecant plane of ϕ will then have degenerated into a surface τ together with the plane corresponding to P. Thus

$$\chi \equiv \tau + \beta. \tag{9.5}$$

To confirm this we consider the net of conics of S_2 represented by χ, taking this to be the net of conics with XYZ as common self-polar triangle, and taking our specialization to be that effected by varying Z continuously so that it comes to lie on XY. Thus, if Z′ is the point $(1, 1, \epsilon)$, then the net of conics with XYZ′ as self-polar triangle has locus-equation

$$\lambda(z - \epsilon x)^2 + \mu(z - \epsilon y)^2 + \nu z^2 = 0$$

and we find that the coordinates a, b, c, f, g, h of conics of this net satisfy the equations $a = -g\epsilon$, $b = -f\epsilon$, $h = 0$; whence, as $\epsilon \to 0$, we see that the locus equations of the specialized net are $a = b = h = 0$. These represent the δ-conics with $z = 0$ as a fixed arm. On the other hand, the envelope equation of the net is

$$\lambda u^2 + \mu v^2 + \nu(u + v + \epsilon w)^2 = 0$$

and the equations satisfied by the dual coordinates A, B, C, F, G, H of conics of this system are $C = \nu\epsilon^2$, $F = \nu\epsilon$, $G = \nu\epsilon$, whence, as $\epsilon \to 0$, the envelope equations of the specialized net are

$C = F = G = 0$. Thus the envelopes of the specialized net are all the point-pairs of the line XY Thus the equivalence relation (9.5) is confirmed.

From (9.4) and (9.5), and by duality, we may now write

$$\tau \equiv \chi - \beta, \qquad \tau' \equiv \chi - \alpha \tag{9.6}$$

and

$$\omega \equiv 2\chi + 2\beta, \quad \omega' \equiv 2\chi + 2\alpha. \tag{9.7}$$

By use of these, in conjunction with our earlier relations (7.5.3)–(7.5.7), we obtain the results

$$\rho \equiv 2\chi + 2\alpha - 2\beta, \quad \sigma \equiv 2\chi - 2\alpha + 2\beta, \tag{9.8}$$

$$\zeta \equiv \zeta' \equiv 2\chi + 2\alpha + 2\beta, \tag{9.9}$$

$$\kappa \equiv \chi + \alpha - \beta, \quad \kappa' \equiv \chi - \alpha + \beta, \tag{9.10}$$

$$\kappa_1 \equiv 2\chi + 2\alpha - 2\beta, \quad \kappa_1' \equiv 2\chi - 2\alpha + 2\beta \tag{9.11}$$

and

$$\kappa_2 \equiv \kappa_2' \equiv 4\chi. \tag{9.12}$$

It will be noticed, in particular, that the surfaces ζ and ζ', though they lie on δ and on η respectively, belong to the same equivalence system $2\chi + 2\alpha + 2\beta$ on Ω; but we shall see shortly that this equivalence system contains surfaces which are irreducible and do not lie on either δ or η.

9.1. *Self-dual doubly infinite systems*

We now make a diversion to consider some of the ∞^2-systems of conics that have the special property of being *self-dual*. A self-dual system of complete conics is one whose image surface on Ω is represented on A_5 and on B_5 respectively by projectively equivalent surfaces F and F′, such that F′ is related to ψ in exactly the same way as F is related to ϕ. Its image surface on Ω will belong to an equivalence system of the form $m(\alpha+\beta)+n\chi$, where m, n are non-negative integers. The simplest example of such a system, of course, is that of conics with a common self-polar triangle, represented on Ω by a surface χ (in A_5 by a trisecant plane of ϕ, and in B_5 by a trisecant plane of ψ). For each of the other systems Σ to be considered we shall write $\Sigma \equiv m(\alpha+\beta)+n\chi$ if $m(\alpha+\beta)+n\chi$ is the equivalence valuation of its image surface on Ω.

A system $\Sigma \equiv \alpha+\beta+\chi$. We define this to be the system of (complete) conics of S_2 such that its model in A_5 is a quadric surface ξ which meets ϕ in a conic k (not meeting ϕ in any other point nor touching ϕ at any point of k). We find then, easily, that its model $\xi' = \tau(\xi)$ in B_5 is also a quadric surface, and that ξ' meets ψ in a conic. Since ξ can degenerate into a conic-plane of ϕ together with a plane meeting ϕ in two points, we deduce that $\Sigma \equiv \alpha+\beta+\chi$ as stated.

A construction for a system Σ of general type is as follows. We choose a triangle XYZ in S_2; then a proper conic g touching ZX and ZY at X and Y respectively, and finally a homographic correspondence (P, P′) on XY with X and Y as united points. Then if U, V are the points where ZP′ meets the tangents to g from P, the pencil of conics that touch PU and PV at U and V respectively generates, as P varies, a system Σ. The generating pencils of conics are mapped on the generators of one system of ξ, and Σ contains a second system of generating pencils, mapped on the generators of the other system on ξ, obtained by interchanging the roles of P and P′. The δ-conics of Σ are the pairs of tangents to g from the points of XY; its η-conics are the point-pairs (U, V), these being the pairs of points in which lines through Z meet a second conic g' touching

ZX and ZY at X and Y; and the two $\delta\eta$-conics of Σ are those with ZX, ZY as axes and X and Y as vertices respectively. The equation of Σ can be taken in the form

$$(x-\mu y)^2+\mu z^2+\nu(x-k\mu y)^2 = 0$$

where μ, ν are the variable parameters and k is the modulus of the homographic correspondence (P, P′).

The special case in which ξ is a quadric cone arises when $k = -1$; and in this case Σ is projectively equivalent to the system of conics with a fixed auxiliary circle.

A system $\Sigma \equiv 2\chi+\alpha+\beta$. We define this to be the system of conics that is represented in A_5 by a rational normal cubic scroll R that meets ϕ in a conic and two further points. We verify then that $\tau(R)$ – the model of R in B_5 – is another cubic scroll R′ which meets ψ in a conic and two further points.

Since this system is of rather less interest in itself, we confine ourselves to giving its reduced locus-equation in the following form:

$$\theta[(a+\lambda)x^2+(b+\lambda)y^2+(c+\lambda)z^2]+\{\alpha(a+\lambda)x+\beta(b+\lambda)y-[\alpha\xi(a+\lambda)+\beta\eta(b+\lambda)]z\}^2 = 0$$

where λ, θ are the variable parameters of the system and a, b, c, α, β, ξ, η are constants. For fixed λ, this equation defines a double-contact pencil, representing a generator of R.

A system $\Sigma \equiv 2\chi+2\alpha+2\beta$. We define this to be a system of conics – general of its kind – that is represented in A_5 by a Veronese surface ε that meets ϕ in a prime section C of this surface. We assume in particular, that C is irreducible and that ε does not lie on the chord-locus M of ϕ. We verify easily, then, that $\tau(\varepsilon)$ – the B_5-model of Σ – is also a Veronese surface, and that this meets ψ in a prime section.

Before proceeding further with this system Σ, we shall prove the following lemma, a result that could not easily have been foreseen.

LEMMA. *If* ε *is a Veronese surface in* A_5 *that meets* ϕ *in one of its prime sections* C (*an irreducible* ${}^0C^4$), *then* ε *is in perspective with* ϕ *from a point of* A_5.

Proof. Let Π be the prime that cuts C on ϕ. Our proof will consist in showing that there exists a homology of A_5, with Π as its axial prime, that carries ϕ into ε, the vertex of this homology being then the required centre of perspective for ϕ and ε.

We begin, then, by remarking that there exist many self-collineations of A_5 that carry ϕ into ε – this is true for every pair of Veronese surfaces in A_5 – and we denote any one of these by ϖ. Further we denote by C_1 the prime section of ϕ that is carried by ϖ into C. If σ is any self-collineation of A_5 that leaves ϕ invariant – briefly a self-collineation of ϕ – then $\varpi\sigma$ also carries ϕ into ε; and we propose to show that σ can be so chosen that $\varpi\sigma$ leaves every point of C invariant. This, namely, would show that $\varpi\sigma$ induces the identical collineation on Π, being therefore a homology of A_5, as was required to be proved.

What is required, then, is to show that σ can be so chosen that it carries each point of C into the point of C_1 that is then returned by ϖ to the same point of C; in other words, we have to be able to choose σ so that it carries C into C_1 and induces a preassigned homography between the points of C and those of C_1, the inverse of that induced from C_1 to C by ϖ.

Consider then the ordinary representation of ϕ on a plane S_2, in which C and C_1 are represented by conics c and c_1 of S_2. We remark (i) that all the self-collineations of ϕ arise directly from self-collineations of S_2 (and conversely), and (ii) that every homographic correspondence between points of c and points of c_1 is induced by a unique self-collineation of S_2 that carries c into c_1. By virtue of (i) and (ii) it follows that there exists a unique self-collineation σ of ϕ that carries C into C_1 and induces the required homography between these two curves; whence, by what we said previously, $\varpi\sigma$ is a homology of A_5. This proves the lemma.

We now return to the system of conics Σ that is represented in A_5 by the Veronese surface ϵ meeting ϕ in the prime section C. By the lemma, it follows that ϕ and ϵ are in perspective from a point V of A_5 (external to both surfaces); so that ϵ lies on the threefold cone $V(\phi)$. If c is the conic of S_2 whose image point is V, then $V(\phi)$ is the A_5-model of the conics of S_2 that have double-contact with c. Hence:

The conics of Σ *all have double-contact with a fixed conic c of* S_2.

Let c be given by the equation

$$c\colon x^2+y^2+z^2 = 0. \tag{9.1.1}$$

We find then that the locus-equation of Σ can be taken to be

$$\Sigma\colon k(\lambda x+\mu y+\nu z)^2+(\alpha\lambda^2+\beta\mu^2+\gamma\nu^2)\,(x^2+y^2+z^2) = 0 \tag{9.1.2}$$

where λ, μ, ν are (homogeneous) variable parameters and k, α, β, γ are constants.

The locus (9.1.2) becomes the repeated line $\lambda x+\mu y+\nu z = 0$ if $\alpha\lambda^2+\beta\mu^2+\gamma\nu^2 = 0$; whence the repeated lines of Σ, axes of its η-conics, are the tangents to the conic c_1 given by

$$c_1\colon \frac{x^2}{\alpha}+\frac{y^2}{\beta}+\frac{z^2}{\gamma} = 0. \tag{9.1.3}$$

Further, the locus (9.1.2) is a line-pair (as distinct from a repeated line) if

$$k(\lambda^2+\mu^2+\nu^2)+\alpha\lambda^2+\beta\mu^2+\gamma\nu^2 = 0,$$

the equation of this line-pair being

$$(\lambda x+\mu y+\nu z)^2-(\lambda^2+\mu^2+\nu^2)\,(x^2+y^2+z^2) = 0.$$

From this it follows that the line-pairs (δ-conics) of Σ are the pairs of tangents to c from the points of the conic c_2 whose equation is

$$c_2\colon (k+\alpha)\,x^2+(k+\beta)\,y^2+(k+\gamma)\,z^2 = 0. \tag{9.1.4}$$

It will be noticed, then, that c_1 touches the tangents to c at the points in which c is met by c_2. The $\delta\eta$-conics of Σ, then, are those with the above four tangents as axes and their points of contact with c as vertices. Finally, to complete the picture, we remark that the point-pair on each tangent to c_1 that makes it into an η-conic of Σ is that in which the tangent meets c.

For any further discussion of the geometry of Σ, we need to investigate the properties of the ∞^2 conics k^1 that lie on ϵ, each of which meets ϕ in two points of C. Each such k^1 maps a quadratic (self-dual) ∞^1-system of conics of Σ of the kind discussed in §8; and we shall denote any such system by σ. Each σ is defined by imposing a linear condition $p\lambda+q\mu+r\nu = 0$ on the parameters λ, μ, ν in equation (9.1.2) of Σ; and (by §8) it is one of the three irreducible systems of conics of S_2 that have double-contact with each of two fixed conics of which one, in the present case, must be c (which has double contact with all the conics of Σ). Let $c(p,q,r)$ be the other conic which has double contact with the conics of σ. We find then that the equation of $c(p,q,r)$ is

$$c(p,q,r)\colon \left(\frac{p^2}{\alpha}+\frac{q^2}{\beta}+\frac{r^2}{\gamma}\right)\left[k\left(\frac{x^2}{\alpha}+\frac{y^2}{\beta}+\frac{z^2}{\gamma}\right)+(x^2+y^2+z^2)\right]-k\left(\frac{px}{\alpha}+\frac{qy}{\beta}+\frac{rz}{\gamma}\right)^2 = 0. \tag{9.1.5}$$

From this it appears that the point (p,q,r) of S_2 is a vertex of the common self-polar triangle of c and $c(p,q,r)$, and that σ consists precisely of the conics that have double-contact with each of c and $c(p,q,r)$ in such a way that their chords of contact both pass through (p,q,r). We now leave it to the reader to investigate further the properties of the ∞^2-system of conics $c(p,q,r)$, a system of the same type as Σ and closely associated with it.

One final remark concerns the equivalence valuation $2\chi+2\alpha+2\beta$ of Σ. This follows, for example, by allowing ϵ to degenerate (in A_5) into a pair of quadric surfaces which each meet ϕ in a conic and have in common a line through the point of intersection of the two conics.

Conics that have four-point contact with a fixed conic s. We comment here only briefly on this system Σ because, as we now show, it is a specialization of the previous case, of the same equivalence valuation $2\chi+2\alpha+2\beta$. Let O be the point of A_5 that maps the fixed conic s. Then the A_5-model of Σ is easily seen to be the cone O(C), where C is the section of ϕ by the polar prime Π of O with respect to the cubic primal M; and this quartic cone is a degenerate Veronese surface through C. The Ω-model of this cone is a surface F^{16}, projective model of a curve system $C^8[O^6, O_1^2, O_2^2, O_3^2]$, where O_1, O_2, O_3 are all in the first neighbourhood of O; and F^{16} meets $\delta\eta$ (abnormally) in a curve of the system $2a+2b$, having no further intersection with δ or η.

A system $\Sigma \equiv 4\chi$. Our last example is that of the ∞^2-system Σ of conics of S_2 whose A_5-model is a Veronese surface ϵ, not lying on M, which touches ϕ at three points. If ϵ is mapped on a plane π by means of all the conics of π, then quadrics Φ through ϕ will meet ϵ in curves mapped on π by quadrics with three fixed double points, a system equivalent over a quadratic transformation of π to the system of all conics of π. Thus $\tau(\epsilon)$ – the B_5-model of Σ – is another Veronese surface; and this touches ψ at three points. Thus Σ is a self-dual system. The Ω-model of Σ is a surface F^{24}, model of a plane curve system $C^6[O_1^2, O_2^2, O_3^2]$; and by allowing ϵ to degenerate into a triad of tangent planes of ϕ together with the plane which meets these in lines, and by using (9.5) and (9.1), we deduce that Σ has the equivalence valuation $3\tau+(\chi+3\beta) \equiv 3(\chi-\beta)+(\chi+3\beta) \equiv 4\chi$.

A system such as Σ can be constructed as follows. Let s be a fixed conic of S_2 and XYZ a fixed triangle; and take Σ to be the ∞^2-system of conics of the form $\varpi(s)$, where ϖ is a variable collineation of S_2 of the form

$$\varpi\colon x' = \lambda x, \quad y' = \mu y, \quad z' = \nu z$$

with X, Y, Z as triad of united points. It is easy to verify, then, that (in general) Σ has, as its A_5-model, a Veronese surface ϵ that touches ϕ at each of three points.

If F^{24} is the Ω-model of Σ, then it may be verified that F^{24} *touches* δ along three conics g_i $(i = 1,2,3)$ and touches η along three conics h_i $(i = 1,2,3)$ such that these two triads of conics are the sets of alternate sides of a closed hexagon of conics whose vertices lie on $\delta\eta$. Each g_i has double contact with a conic a_i $(i = 1,2,3)$ of $\delta\eta$ at a pair of vertices of the hexagon; and, similarly, each h_i has double contact with a conic b_i $(i = 1,2,3)$ at a pair of vertices of the hexagon.

Now consider the special case of the above construction for which s is taken to be a line-pair. The resulting family of conics – say $\bar{\Sigma}$ – consists entirely of δ-conics; so that it is no longer self-dual. Further, it can now be characterized in another way, namely as the aggregate of pairs of lines that correspond in a *fixed* collineation ϖ with X, Y, Z as united points. Thus $\bar{\Sigma}$ is a system κ_2 that was previously discussed by us ((iv) of § 7.5) in its role as an ∞^2 system of δ-conics. Thus the equivalence system $2\chi+2\alpha+2\beta$ of doubly infinite systems of conics contains members (irreducible systems) of which some are contained in δ, others in η, and still others neither in δ nor in η.

10. Triply infinite systems of complete conics

We are concerned here essentially with threefolds on Ω, for which, as noted in §7.3, the triad μ^2, $\widehat{\mu\nu}$, ν^2 is a complete equivalence base. However, in connection with equivalence bases on δ and on η, we were led to consider two other important threefolds on Ω, namely those which we denoted by J and J′, of which J lies on δ and maps δ-conics with vertices on a fixed line of S_2 while J′ lies on η and maps η-conics with axes through a fixed point. Further, we proved in §7.4 that each of J, J′ is a rational scrollar variety ${}^0R_3^9$ of general type, generated in the first case by planes α and in the second by planes β (joins, in each case, of corresponding points of homographic ranges on a conic, a twisted cubic and a rational normal quartic curve). In A_5, J is represented by the cubic cone of conic-planes of ϕ that pass through a fixed point of this surface, while J′ is represented by the total neighbourhood of a conic of ϕ.

We observe first, then, that when a general solid of A_5, representing a threefold μ^2 on Ω, is specialized into a solid which meets ϕ in a conic k, then μ^2 is specialized into a threefold $\widehat{\mu\nu}$ together with the threefold J′ that corresponds to the total neighbourhood of k in A_5. With duality, this gives the pair of formulae

$$\mu^2 \equiv \widehat{\mu\nu} + J' \quad \text{and} \quad \nu^2 \equiv \widehat{\mu\nu} + J. \tag{10.1}$$

Further, as already noted in §7.4, the neighbourhood of k on M represents a surface σ (cf. §2.2), the intersection of $\delta\eta$ with J′; whence, and by duality, we have (on Ω) the equivalence relations

$$J'\delta \equiv \sigma \quad \text{and} \quad J\eta \equiv \rho. \tag{10.2}$$

Recalling now that $\mu\nu \equiv 2\widehat{\mu\nu}$, we have the virtual equivalence formula

$$\delta\eta \sim (2\nu - \mu)(2\mu - \nu) \sim -2\mu^2 + 10\widehat{\mu\nu} - 2\nu^2 \tag{10.3}$$

and from this and (10.1) we deduce the (virtual) relations

$$\left.\begin{aligned} 6\mu^2 &\sim \delta\eta + 2J + 8J', \\ 6\widehat{\mu\nu} &\sim \delta\eta + 2J + 2J', \\ 6\nu^2 &\sim \delta\eta + 8J + 2J', \end{aligned}\right\} \tag{10.4}$$

which relate $\delta\eta$, J and J′ to μ^2, $\widehat{\mu\nu}$ and ν^2.

From (10.4) we deduce the relations

$$2\mu\delta \sim \delta\eta + 2J \quad \text{and} \quad 2\nu\eta \sim \delta\eta + 2J'; \tag{10.5}$$

and these lead again to (7.4.2) and its dual, namely

$$\nu\delta \equiv 2J \quad \text{and} \quad \mu\eta \equiv 2J'. \tag{10.6}$$

We come now to the second objective of this section which is to exhibit the internal structure of each of a series of important threefold systems of complete conics. Thus, for example, if Σ is the system of complete conics that is mapped on Ω by a general threefold μ^2, then we would aim to give some description of μ^2 and of its degeneration varieties – the surfaces $\mu^2\delta$ and $\mu^2\eta$ and the curve $\mu^2\delta\eta$ – together with some indication of the inter-relations of these varieties. As most of these investigations of threefold systems are somewhat long and specialized, we shall give at this point only accounts of a few typical cases, relegating the others to an appendix (Appendix A) for the record. We start now with:

The system $\Sigma \equiv \mu^2$. In this case, as in most similar cases, we first draw up, as preliminary information, a table of the *virtual* orders, computed by the method explained in § 6, of the relevant varieties on Ω. If, as may happen, some of the varieties are multiple intersections, the corresponding entries in the table have to be interpreted by detailed examination. In the present case the table for μ^2 is as follows:

	μ^2	$\mu^2\delta$	$\mu^2\eta$	$\mu^2\delta\eta$
virtual order	23	19	4	8

We observe, now, that μ^2 maps the complete conics of S_2 that are apolar to a general pair of conic-envelopes; and it is represented in A_5 by a general solid of A_5, say T, which meets ϕ in four points $P_1, \ldots, P_4$. The cubic primals of $|K|$ meet T in the complete system of cubic surfaces of T that pass through $P_1, \ldots, P_4$; and hence:

The general threefold μ^2 *is a* $V_3^{23}[15]$, *projective model of the complete system of cubic surfaces through four points of a solid* T.

The intersection $\mu^2\delta$, being represented in A_5 by the four-nodal cubic surface in which T meets M, is found to be a surface F_2^{19}, projective model of the system of plane quartic curves through the six vertices of a complete quadrilateral. The surface $\mu^2\eta$ consists of the four planes β that represent the neighbourhoods of $P_1, \ldots, P_4$ in A_5; and finally the curve $\mu^2\delta\eta$ consists of the four conics b that lie in the four planes β in question. In the plane representation of F_2^{19}, the four conics b correspond to the sides of the aforementioned complete quadrilateral.

In Appendix A, we give discussions analogous to the above, of the systems of complete conics that are mapped on Ω by the following threefolds: (i) $\bar{\mu}^2$, the model of complete 'circles' of S_2; (ii) the variety $\mu\nu$; (iii) the variety $\bar{\mu}\bar{\nu}$ (conics that pass through a fixed point and touch a fixed line, not through the point); (iv) the threefold $\widehat{\mu\nu}$; (v) the threefold $\overline{\mu\nu}$ (conics that touch a given line at a given point).

Conics that have double-contact with a fixed conic g. Let D denote this system of conics and let d denote its threefold image on Ω. Further let d^* be the A_5-model of D; and let G and G* be the image points of g in Ω and in A_5 respectively. Then d^*, as previously noted, is the quartic threefold cone $G^*(\phi)$; and this implies that d is simply generated by the ∞^2 conics on Ω (representing pencil-ranges) that pass through G.

Since μ^2, $\widehat{\mu\nu}$ and ν^2 are a base for threefolds on Ω, we may write

$$d \equiv p\mu^2 + q\widehat{\mu\nu} + r\nu^2 \tag{10.7}$$

where p, q, r are integers; and further, since D is plainly a self-dual system, we shall have $r = p$. To evaluate $p(=r)$ and q, we multiply (10.7) symbolically by α and by χ in turn and then evaluate directly (as numbers) the condition products that arise. We find readily that

$$d\alpha = d\chi = 1, \quad \alpha(\mu^2, \widehat{\mu\nu}, \nu^2) = (1, 0, 0) \quad \text{and} \quad \chi(\mu^2, \widehat{\mu\nu}, \nu^2) = (1, 1, 1);$$

and from these values there result two equations for $p(=r)$ and q which give $p = r = 1$ and $q = -1$. Hence

$$d \sim \mu^2 - \widehat{\mu\nu} + \nu^2. \tag{10.8}$$

With the permissible substitution of $\frac{1}{2}\mu\nu$ for $\widehat{\mu\nu}$ and the use of (6.5), we find that

$$d\mu^3 = (\mu^2 - \widehat{\mu\nu} + \nu^2)\,\mu^3 = 4$$

in agreement with the fact that there are in general four conics that pass through three given points and have double contact with a fixed conic.

Or again

$$d^2\mu = (\mu^2 - \tfrac{1}{2}\mu\nu + \nu^2)^2\mu = 6,$$

in agreement with the fact that there are in general six conics that have double contact with each of two fixed conics and pass through a given point.

The relation (10.8) can also be obtained directly by specializing g to a proper line-pair $\bar{g}$ (a δ-conic), in which case the resulting specialization of the system D consists of

(i) the ∞^3-system of conics that touch the arms of $\bar{g}$, and

(ii) the ∞^3-system of η-conics whose axes pass through the vertex of $\bar{g}$.

The former of these systems is mapped on Ω by a variety $\bar{\nu}^2 \equiv \nu^2$ and the latter is mapped on a threefold J′; and this gives the relation

$$d \equiv \nu^2 + J'$$

on Ω. Since, by (10.1), $J' \equiv \mu^2 - \widehat{\mu\nu}$, this gives $d \sim \mu^2 - \widehat{\mu\nu} + \nu^2$, which is (10.8).

The δ-conics of D are the pairs of tangents to g, a system mapped on Ω by a surface ζ (cf. §7.5 (i)); and dually the η-conics of D are mapped on a surface ζ'; i.e.

$$d\delta \equiv \zeta \quad \text{and} \quad d\eta \equiv \zeta'. \tag{10.9}$$

Finally, the $\delta\eta$-conics of D are mapped on a rational octavic curve of $\delta\eta$; in fact

$$d\delta\eta \equiv 2a + 2b. \tag{10.10}$$

Conics that have three-point contact with a fixed conic g. Let E denote this (self-dual) system of conics; let ϵ denote its model threefold on Ω; and let G and G_0 denote respectively the image points of g on Ω and in A_5.

Since the conics that have three-point contact with g at a fixed point P form the linear net through three consecutive points P, P_1, P_2 of g, it is convenient to consider first in general the second order curve elements $\mathscr{E}_2$ of S_2, sets of three consecutive points of S_2. The conics that contain any such $\mathscr{E}_2$ form a linear net, a specialization of that mapped in A_5 by a trisecant plane of ϕ. It is known, then, that the specialized net is mapped in A_5 by an 'osculating plane' of ϕ, i.e. by a plane that contains a triad of consecutive points of ϕ (an $\mathscr{E}_2$ of ϕ); also that there exists thereby a birational correspondence between the ∞^4-aggregate of $\mathscr{E}_2$ in S_2 and the ∞^4-aggregate of osculating planes of ϕ (cf. Semple (1954), pp. 25 and 32). Thus the system E, being generated by ∞^1 of the special linear nets, each containing g itself, must have as its A_5-model the threefold generated by the ∞^1 osculating planes of ϕ that pass through G_0. We now find readily, however, that any osculating plane of ϕ that passes through G_0 is the join of G_0 to a tangent line of the curve C in which ϕ is met by the polar prime Π of G_0, with respect to the cubic primal M of A_5. We assert then:

The A_5-model of E *is the sextic threefold cone* $G_0(T)$, *where* T *is the (sextic) surface of tangents to the curve* C, *section of* ϕ *by the prime* Π.

Since each plane joining G_0 to a tangent t of C meets ϕ in two consecutive points of ϕ on t and in a third point of ϕ consecutive to these two, it follows that C must count triply as a curve of intersection of $G_0(T)$ with ϕ. This can be verified analytically.† Thus, since C is equivalent on

† The fourfold cone projecting $G_0(T)$ from a *generic* point of A_5 meets ϕ (normally) in a curve of which C is a triple component.

ϕ to a pair of conics, it can be argued that $G_0(T)$ is equivalent, *in* A_5 *and relative to* ϕ, to a combination of six solids of A_5 that each meet ϕ in a conic, and hence that its transform ϵ on Ω will be equivalent on Ω to $6\widehat{\mu\nu}$. Thus

$$\epsilon \sim 6\widehat{\mu\nu}. \tag{10.11}$$

We may remark, in passing, that the Ω-model of conics that have three-point contact with g at a fixed point is a surface χ^* (a specialization of the general χ) that is the projective model of a system of plane cubics with three consecutive (but not collinear) base points. Each such χ^* possesses therefore a binode. This lies on $\delta\eta$; and the six intersections of the general χ with $\delta\eta$ coincide for χ^* in this binode. The δ-conics of E are those for which one arm touches g while the other passes through its point of contact, and we find, by a degeneration of g, that their Ω-model is a surface of the equivalence system $2\tau+4\alpha \equiv 2\chi-2\beta+4\alpha$ (of order 14) on δ. Also, dually, the η-conics of E have an Ω-model of the system $2\chi-2\alpha+4\beta$; and the $\delta\eta$-conics of E are represented on Ω by a curve of the equivalence system $2a+2b$.

To confirm the equivalence valuation (10.11) of ϵ, we now proceed, as in the previous case, by computing p and q such that

$$\epsilon \sim p\mu^2 + q\widehat{\mu\nu} + p\nu^2.$$

To do this we multiply the above equation symbolically by α and χ in turn, using our previous evaluations $\alpha(\mu^2, \widehat{\mu\nu}, \nu^2) = (1, 0, 0)$ and $\chi(\mu^2, \widehat{\mu\nu}, \nu^2) = (1, 1, 1)$, so that only $\alpha\epsilon$ and $\chi\epsilon$ remain to be evaluated. Since a δ-conic (or $\delta\eta$-conic) cannot belong to E if its vertex does not lie on g, we deduce that $\alpha\epsilon = 0$; from which it follows that $p = 0$. Dually, also, we will have $\beta\epsilon = 0$. Further, since $\chi\epsilon = (\chi+3\beta)\epsilon = \mu^3\epsilon = \bar{\mu}^3\epsilon$ it follows that $\chi\epsilon$ is equal to the number of conics that have three-point contact with g and pass through three given points; and this number is six, the number of inflexional lines of a trinodal plane quartic curve. This gives $q = 6$, whence $\epsilon \sim 6\widehat{\mu\nu}$ as stated in (10.11).

We now note finally that, by use of (10.11), the computed *virtual* orders of the surfaces $\epsilon\delta$ and $\epsilon\eta$ are each found to be 42; whence the surfaces – each of order 14 – which map the δ-conics and the η-conics of E must count triply in $\epsilon\delta$ and in $\epsilon\eta$ respectively. Similarly, since the computed order of the curve $\epsilon\delta\eta$ is found to be 48, it follows that the octavic curve, of the equivalence family $2a+2b$ on Ω, that maps the $\delta\eta$-conics of E must count sixfold in the intersection $\epsilon\delta\eta$.

11. The method of degenerate collineations

We now come back to the general problem of solving, in a definitive way, outstanding questions relating to the discovery of minimum algebraic equivalence bases for sub-varieties on Ω and on its subordinate varieties δ, η and $\delta\eta$. The general method to be used – referred to usually as the *method of degenerate collineations* – is that originally devised by van der Waerden, and applied by him in particular to solve the problems of bases for fourfolds and threefolds on Ω [W, §§2, 3]. In what follows we shall first give a detailed analysis of van der Waerden's method and results, and we shall then apply his methods, in so far as may be necessary, to solve other outstanding problems of the base, in particular to those on the model varieties subordinate to Ω.

As regards our previous remarks on the present problems, we recall

(i) that bases for curves and surfaces on $\delta\eta$ are already known (cf. §2.2);

(ii) that, in §§7.1–7.4, we put forward prospective bases for fourfolds and threefolds on Ω; and then, on the basis of assumptions (*a*) and (*b*) of §7, we selected complementary bases for curves and surfaces on Ω, verifying the non-singularity of the two relevant intersection matrices;

(iii) that, for geometry on δ, we first chose a prospective base for threefolds; and then, again on the basis of assumptions (a) and (b), we selected prospective bases for curves and surfaces on δ, and again verified the non-singularity of the two relevant intersection matrices; and

(iv) that bases on η will follow from those for δ by duality. Essentially, then, what remains to be done is to confirm that the proposed bases for fourfolds and threefolds on Ω, as also those for threefolds and surfaces on δ, are in fact complete minimal bases; and we do this by using the method of degenerate collineations.

Suppose then that Σ_d is any irreducible system of complete conics of S_2, of dimension d $(1 \leqslant d \leqslant 4)$. We shall use the same symbol Σ_d to denote also the model of the system, whether this be envisaged as a one-way d-fold on Ω or as its abstractly equivalent two-way d-fold on the two-way model $\mathscr{W}$ of complete conics (cf. §2). We base operations on a set of equations that completely define Σ_d; and we take these for convenience to be its two-way equations in the form

$$f_\lambda(a_{00}, a_{01}, a_{11}, a_{02}, a_{12}, a_{22}; b_{00}, b_{01}, b_{11}, b_{02}, b_{12}, b_{22}) = 0 \quad (\lambda = 1, \ldots, s), \tag{11.1}$$

these to include, of course, the two-way equations of $\mathscr{W}$ (namely (2.1)). We suppose now that we have chosen a one-parameter family of self-collineations of S_2, to be denoted by $\varpi(\epsilon)$, where ϵ is the parameter, such that $\varpi(1)$ is the identity collineation, $\varpi(\epsilon)$ is non-singular if $\epsilon \neq 0$, while $\varpi(0)$ is a singular collineation. For $\epsilon \neq 0$, $\varpi(\epsilon)$ carries Σ_d into a system $\Sigma_d(\epsilon)$; and, as $\epsilon \to 0$, $\Sigma_d(\epsilon)$ (as is well known) will tend to a well defined system $\Sigma_d(0)$ – possibly reducible – which is by definition algebraically equivalent to Σ_d. We now wish to find what can be said about the irreducible component systems of $\Sigma_d(0)$; and to do this we proceed as follows.

First, by use of the equations of $\varpi(\epsilon)$, we form the equations (analogous to (11.1)) of $\Sigma_d(\epsilon)$; and we then denote by (E) the set of equations derived from those of $\Sigma_d(\epsilon)$ by putting $\epsilon = 0$. From the general theory of algebraic correspondences (see, for example, Hodge & Pedoe 1952, ch. XI) we know that the equations (E), though they will not in general suffice to define $\Sigma_d(0)$, will nevertheless be satisfied by the conics of $\Sigma_d(0)$; and they give us therefore useful information about the irreducible component systems of $\Sigma_d(0)$.

Before proceeding to apply the above method, we recall a result, known as the *moving lemma*, of which we shall be making use in what follows. A detailed statement and proof of the general form of this result are set out on pp. 190–191 of Hodge & Pedoe (1952). The content of the lemma, as we apply it here, can be stated as follows:

THE MOVING LEMMA. *Let* V_n *be a non-singular algebraic variety, and let* V_a *and* V_b *be effective sub-varieties of* V_n *such that* $a+b \geqslant n$, *and* V_b *is irreducible. Further let the (point-set) intersection* $V_a \wedge V_b$ *be of abnormal dimension (i.e. of dimension* $> a+b-n$*) for sub-varieties of* V_n. *(This includes in particular the case in which* V_a *is contained in* V_b.*) Then there exists on* V_n *a virtual variety* U_a, *a difference* $U_a^{(1)} - U_a^{(2)}$ *of effective varieties, such that* (i) $U_a \sim V_a$ *(on* V_n*), and* (ii) *each of* $U_a^{(1)}$ *and* $U_a^{(2)}$ *meets* V_b *simply in a variety of normal dimension* $a+b-n$.

In any application of the method of degenerate collineations, as we have described it above, the choice of the system $\varpi(\epsilon)$ – the *reducing system* as we may call it – will obviously be important; and in fact we shall have to use different types of reducing system for different objectives. We now consider the effect of our first choice of such a reducing system which we shall denote by $\varpi_1(\epsilon)$.

11.1. *The reducing system* $\varpi_1(\epsilon)$

This is the system† of S_2-collineations, whose equations, as they apply to points and lines of S_2, are as follows:

$$\varpi_1(\epsilon):\begin{array}{l} x_0' = x_0, \quad x_1' = \epsilon x_1, \quad x_2' = \epsilon^2 x_2, \\ u_0' = \epsilon^2 u_0, \quad u_1' = \epsilon u_1, \quad u_2' = u_2. \end{array} \tag{11.1.1}$$

We find, then, that the coordinate vector of the complete conic that is transformed by $\varpi_1(\epsilon)$ into the general complete conic (a, b) is

$$(a_{00}, \epsilon a_{01}, \epsilon^2 a_{11}, \epsilon^2 a_{02}, \epsilon^3 a_{12}, \epsilon^4 a_{22};\ \epsilon^4 b_{00}, \epsilon^3 b_{01}, \epsilon^2 b_{11}, \epsilon^2 b_{02}, \epsilon b_{12}, b_{22})$$

and from this it follows that the equations of $\Sigma_d(\epsilon)$, for $\epsilon \neq 0$, are

$$\Sigma_d(\epsilon): f_\lambda(a_{00}, \epsilon a_{01}, \epsilon^2 a_{11}, \epsilon^2 a_{02}, \epsilon^3 a_{12}, \epsilon^4 a_{22};\ \epsilon^4 b_{00}, \epsilon^3 b_{01}, \epsilon^2 b_{11}, \epsilon^2 b_{02}, \epsilon b_{12}, b_{22}) = 0 \quad (\lambda = 1, \ldots, s) \tag{11.1.2}$$

From these, then, by putting $\epsilon = 0$, we find the set of equations

$$(\mathrm{E}): f_\lambda(a_{00}, 0, 0, 0, 0, 0;\ 0, 0, 0, 0, 0, b_{22}) = 0 \quad (\lambda = 1, \ldots, s) \tag{11.1.3}$$

that will be satisfied, by what we have said above, by the limiting system $\Sigma_d(0)$ of $\Sigma_d(\epsilon)$ as $\epsilon \to 0$. These equations (E) are all of the form

$$k_\lambda a_{00}^{M_\lambda} b_{22}^{N_\lambda} = 0 \quad (\lambda = 1, \ldots, s), \tag{11.1.4}$$

where the k_λ are constants and the M_λ, N_λ are non-negative integers.

If all the k_λ were zero, this would mean that the equations (11.1) of Σ_d are satisfied by the coordinate vector $(1, 0, 0, 0, 0, 0;\ 0, 0, 0, 0, 0, 1)$; in other words, that Σ_d contains the $\delta\eta$-conic with $x_0 = 0$ as axis and the reference point X_2 as vertex. Let it be assumed, for the moment, that Σ_d does not contain all $\delta\eta$-conics. Then the reference system in S_2 can be so chosen that Σ_d does not contain the particular $\delta\eta$-conic in question. We may therefore assume that one or more of the k_λ will not be zero. It now follows from (11.1.4) that every conic of $\Sigma_d(0)$, and in particular the generic member of every irreducible component of the system $\Sigma_d(0)$, must satisfy one or other of the equations $a_{00} = 0$ and $b_{22} = 0$; i.e. it must either pass through the fixed point X_0 or touch the fixed line X_0X_1. In other words, each irreducible component of the system $\Sigma_d(0)$ is either contained in a system $\bar{\mu}$ $(\equiv \mu)$ or in a system $\bar{\nu}$ $(\equiv \nu)$.

If $d = 4$, and if Σ_4 does not contain the threefold system $\delta\eta$, the above establishes an equivalence relation of the form

$$\Sigma_4 \equiv m\mu + n\nu \tag{11.1.5}$$

where m and n are non-negative integers. Further if Σ_4 contains $\delta\eta$, then the moving lemma (§11) tells us that $\Sigma_4 \sim \Sigma_4' - \Sigma_4''$ where each of the effective varieties Σ_4' and Σ_4'' does not contain $\delta\eta$. Thus in all cases we have a *virtual* equivalence of the form

$$\Sigma_4 \sim m\mu + n\nu, \tag{11.1.6}$$

when m and n are integers (positive, negative or zero). Hence:

A pair (μ, ν) *is a complete equivalence base for fourfolds on* Ω.

† The discovery and properties of this useful variant $\varpi_1(\epsilon)$ of van der Waerden's reducing system $\varpi_2(\epsilon)$ are due to Dr J. A. Tyrrell who told me about them many years ago.

While the reducing system $\varpi_1(\epsilon)$ thus gives us a definite result for fourfold systems of complete conics, it only tells us that any system Σ_d with $d \leqslant 3$ is equivalent to a system Σ'_d that lies on a variety $\bar{\mu}$ or $\bar{\nu}$ of Ω. For systems Σ_3 we follow van der Waerden in using a different approach system $\varpi_2(\epsilon)$ as follows.

11.2. *The reducing system* $\varpi_2(\epsilon)$

This is the system of S_2-collineations with equations

$$\varpi_2(\epsilon): \begin{matrix} x'_0 = x_0, & x'_1 = x_1, & x'_2 = \epsilon x_2, \\ u'_0 = \epsilon u_0, & u'_1 = \epsilon u_1, & u'_2 = u_2. \end{matrix} \tag{11.2.1}$$

This has the character, as $\epsilon \to 0$, of a 'gradual projection' of S_2 from the vertex X_2 on to the line X_0X_1. We now find that the complete conic that is transformed by $\varpi_2(\epsilon)$, with $\epsilon \neq 0$, into the general complete conic (a, b) is that with coordinate vector

$$(a_{00}, a_{01}, a_{11}, \epsilon a_{02}, \epsilon a_{12}, \epsilon^2 a_{22}; \epsilon^2 b_{00}, \epsilon^2 b_{01}, \epsilon^2 b_{11}, \epsilon b_{02}, \epsilon b_{12}, b_{22});$$

whence the equations of $\Sigma_3(\epsilon)$, for $\epsilon \neq 0$, take the form

$$\Sigma_3(\epsilon): f_\lambda(a_{00}, a_{01}, a_{11}, \epsilon a_{02}, \epsilon a_{12}, \epsilon^2 a_{22}; \epsilon^2 b_{00}, \epsilon^2 b_{01}, \epsilon^2 b_{11}, \epsilon b_{02}, \epsilon b_{12}, b_{22}) = 0 \quad (\lambda = 1, \ldots, s). \tag{11.2.2}$$

Thus, in this case, putting $\epsilon = 0$, we obtain the equations

$$(\mathrm{E}): f_\lambda(a_{00}, a_{01}, a_{11}, 0, 0, 0; 0, 0, 0, 0, 0, b_{22}) = 0 \quad (\lambda = 1, \ldots, s), \tag{11.2.3}$$

which must be satisfied by every conic of the limiting system $\Sigma_3(0)$ of $\Sigma_3(\epsilon)$ as $\epsilon \to 0$.

If these equations (E) were all nugatory, this would mean that the equations (11.1) of Σ_3 would be satisfied by every coordinate vector of the form

$$(a_{00}, a_{01}, a_{11}, 0, 0, 0; 0, 0, 0, 0, 0, b_{22}),$$

i.e. Σ_3 would contain all δ-conics with X_2 as vertex. If Σ_3 does not consist entirely of δ-conics, then we can certainly choose X_2 so that the above possibility does not arise. Further, X_2 may additionally be chosen so that at most a finite number of δ-conics of Σ_3 have vertex at X_2, and we may suppose that these δ-conics of Σ_3, with X_2 as vertex, are δ_i $(i = 1, \ldots, t)$.

Since the equations (E) are not all nugatory (for X_2 chosen as above), they imply, for any conic (a, b) of $\Sigma_3(0)$, that

either (i) $b_{22} = 0$, in which case (a, b) touches X_0X_1,
or (ii) $b_{22} \neq 0$, in which case the δ-conic $(a_{00}, a_{01}, a_{11}, 0, 0, 0; 0, 0, 0, 0, 0, b_{22})$ belongs to the system Σ_3 and is therefore one of the conics δ_i with X_2 as vertex.

In the latter case the conic (a, b) of $\Sigma_3(0)$ meets X_0X_1 in the same two points as the δ_i in question. It follows, then, from this that every conic of the system $\Sigma_3(0)$, and in particular the generic member of any irreducible component of $\Sigma_3(0)$, either touches X_0X_1 or meets this line in one of a finite set of point-pairs. From this we deduce an equivalence relation of the form

$$\Sigma_3 \equiv m\mu^2 + T_3, \tag{11.2.4}$$

where m is a non-negative integer and T_3 is an effective threefold system of conics which all touch the line X_0X_1. If Σ_3 consists entirely of δ-conics, then, by the moving lemma, it is virtually equivalent to a difference $\Sigma'_3 - \Sigma''_3$, such that neither Σ'_3 nor Σ''_3 consists entirely of δ-conics; and, in place of (11.2.4), we get a virtual equivalence relation of the form

$$\Sigma_3 \sim m\mu^2 + T_3, \tag{11.2.5}$$

where m is an integer (possibly negative), and T_3 may now have positive or negative components, each of which, however, has the line X_0X_1 as a base tangent.

From (11.2.5) we also have, by S_2-duality, an equivalence relation of the form

$$\Sigma_3 \sim n\nu^2 + U_3, \tag{11.2.6}$$

where n is an integer and U_3 is a virtual system such that each of its irreducible components (positive or negative) consists of conics which all pass through a fixed point. We take this point to be X_0.

We now fix attention on (11.2.6) and suppose that U_3^* is any one of the irreducible components of U_3 (whether it enters positively or negatively into U_3). Then U_3^* is a system with X_0 as a base point, so that one of its equations is $a_{00} = 0$. By operating on U_3^* with $\varpi_2(\epsilon)$ exactly as before, and supposing that U_3^* has at most a finite number, t say, of δ-conics with vertex X_2, we are led to a limit system $U_3^*(0)$, equivalent to U_3^*, such that, for every conic (a, b) of $U_3^*(0)$,

either (i) $b_{22} = 0$, so that the conic touches X_0X_1, necessarily at X_0,

or (ii) $b_{22} \neq 0$, in which case the conic meets X_0X_1 in one of a finite set of point-pairs of the form (X_0, P_i), $i = 1, \ldots, t$.

Since the system of conics (earlier denoted by $\overline{\mu\nu}$) that touch a given line at a given point is a specialization of the general system $\widehat{\mu\nu}$, the above analysis leads to an equivalence relation of the form $U_3^* \sim r\widehat{\mu\nu} + s\mu^2$, where r and s are non-negative integers; also, by use of the moving lemma, we can extend the scope of such a relation to include systems U_3^* that consist entirely of δ-conics in the form

$$U_3^* \sim r\widehat{\mu\nu} + s\mu^2, \tag{11.2.7}$$

where the integers r, s may now be positive, negative or zero.

Finally, combining (11.2.7) with (11.2.6), we arrive at the general equivalence relation, applicable to all systems Σ_3, in the form

$$\Sigma_3 \sim m\mu^2 + r\widehat{\mu\nu} + n\nu^2 \tag{11.2.8}$$

where m, r, n are integers. Hence:

The triad $(\mu^2, \widehat{\mu\nu}, \nu^2)$ *constitutes a complete equivalence base for threefolds on* Ω.

Although the above proof applies, with minor adjustments, to the particular threefold δη, we observe directly that

$$\delta\eta \sim (2\nu - \mu)(2\mu - \nu) \sim -2\mu^2 + 5\mu\nu - 2\nu^2 \sim -2\mu^2 + 10\widehat{\mu\nu} - 2\nu^2.$$

11.3. *Equivalence base for threefolds on* δ

The two types, $\varpi_1(\epsilon)$ and $\varpi_2(\epsilon)$, of reduction system that we have been using have the obvious property that the S_2-collineations comprising them all carry any irreducible system of δ-conics into another system of δ-conics, this term being used in the broad sense; and they can therefore be applied to investigate equivalence bases on δ itself as the non-singular model for the δ-conic as an independent geometric variable. The equations of an irreducible system of δ-conics may still be written in the form (11.1), these to include of course the equations of δ itself.

Consider first, then, an irreducible threefold system Σ_3 of δ-conics, other than δη itself; and apply the reduction system $\varpi_1(\epsilon)$ as in § 11.1. This gives, as the limit of $\Sigma_3(\epsilon)$ as $\epsilon \to 0$, an equivalent system $\Sigma_3(0)$ such that every conic (a, b) of $\Sigma_3(0)$ satisfies one or other of the equations $a_{00} = 0$ or $b_{22} = 0$; but now, since (a, b) is a δ-conic (possibly a δη-conic), the condition $b_{22} = 0$ means that the vertex of (a, b) lies on the line X_0X_1. It follows, then, that every irreducible component of $\Sigma_3(0)$ is either the system of δ-conics that pass through X_0 or the system of δ-conics with vertices on X_0X_1. Thus, in the notation of §7.4, we may write

$$\Sigma_3 \equiv m\mathrm{K} + n\mathrm{J}$$

where m and n are non-negative integers. As regards the case where Σ_3 is δη, we recall from §7.4 that $\mathrm{K} \equiv \mu\delta$ and (by (7.4.3)) $\nu\delta = 2\mathrm{J}$ on δ; whence we may write

$$\eta\delta \sim (2\mu - \nu)\delta \sim 2\mathrm{K} - 2\mathrm{J}$$

on δ. It follows, then, without exception, that K *and* J *constitute a* (*complete*) *base for threefolds on* δ.

11.4. *Base for surfaces on* δ

Let Σ_2 denote an irreducible ∞^2-system of δ-conics with equations (11.1); and let us suppose at first that Σ_2 does not consist entirely of δη-conics. We can then choose the reference point X_2 in S_2 so that at most a finite number of members of Σ_2 have vertex at X_2 and that no one of these is a δη-conic. We now apply the reduction system $\varpi_2(\epsilon)$, as in §11.2; and this leads to a system $\Sigma_2(0)$, equivalent to Σ_2, such that every member (a, b) of $\Sigma_2(0)$ – whether a δ-conic or a δη-conic – *either* has $b_{22} = 0$, in which case it has its vertex on X_0X_1, *or* it has $b_{22} \neq 0$, in which case it meets X_0X_1 in one of a finite set of point-pairs on this line. It follows that each irreducible component of $\Sigma_2(0)$ is *either* contained in the ∞^3-system J of δ-conics with vertex on X_0X_1, *or* it is an ∞^2-system of δ-conics that meet X_0X_1 in a fixed pair of points. An irreducible system of the latter kind, however, is either the system τ of δ-conics with X_0X_1 as a fixed arm, in which case it is contained in J, or it is a system ω of δ-conics whose arms each contain one of the two fixed points. Thus we may write

$$\Sigma_2 \equiv m\omega + \Xi_2, \tag{11.4.1}$$

where m is an integer and Ξ_2 is an ∞^2-system of δ-conics (or δη-conics) that is contained in a system J (of δ-conics with vertices on a fixed line).

If Σ_2 consists entirely of δη-conics, then an application of the moving lemma will replace (11.4.1) by the virtual equivalence formula

$$\Sigma_2 \sim m\omega + \Xi_2, \tag{11.4.2}$$

where Ξ_2 is now a virtual ∞^2-system belonging to J.

For the further discussion of Ξ_2, it is possible to proceed as before by considering any one of the irreducible components of Ξ_2 (whether it appears positively or negatively in Ξ_2) and applying to it a reducing system $\varpi_3(\epsilon)$, the inverse of $\varpi_2(\epsilon)$, which has the character of a retraction of S_2 towards X_2. As it happens, however, it is easy to identify a base for surfaces on the threefold J, regarded now as a threefold on δ; for J is, in fact, a rational planar threefold ${}^0R_3^9$ of the general kind generated by planes joining homographically related points of a conic c^2, a twisted cubic c^3 and a rational normal quartic curve c^4, these three spanning a space S_{11} (§7.4). The generating planes of J are α-planes, each representing the δ-conics with vertex at a fixed point of X_0X_1; and J possesses a unique minimum directrix surface τ – an ${}^0R_2^5$ with c^2 and c^3 as directrix curves – representing δ-conics with X_0X_1 as one arm. A base for surfaces on J consists of a generating plane α together with the surface τ; and it follows then that

$$\Xi_2 \sim n\alpha + p\tau \quad (n, p \text{ integers})$$

on J, and therefore also on δ. Combining this with (11.4.2) we have the general formula

$$\Sigma_2 \sim m\omega + n\alpha + p\tau \tag{11.4.3}$$

on δ, where m, n, p are integers. Thus:

The surfaces α, τ, ω *constitute a complete base*† *for surfaces on* δ.

We add a final remark about the threefold J. This concerns the surface ρ – an ${}^3F_2^{12}$ – which maps all the δη-conics with vertices on X_0X_1. This meets each generating plane α of J in the conic a that lies in that plane; and it must therefore be algebraically equivalent (on J and on δ) to a combination of the form $2\tau + n\alpha$. From the orders of the surfaces involved we deduce that $n = 2$; so that

$$\rho \equiv 2\tau + 2\alpha \tag{11.4.4}$$

on δ; and this supplements (7.5.3) which gives the equivalence of σ on δ.

Appendix A. Further Ω-models of triply infinite systems of conics

A 1. *The model* $\bar{\mu}^2$ *of (complete) conics through two fixed points*

This can be thought of as the Ω-model of all circles in S_2; and it is perhaps surprising that its investigation turns out to be one of the most difficult that we shall encounter.

We begin then by comparing $\bar{\mu}^2$ with the general threefold μ^2 that has already been discussed in some detail (§10). Each of μ^2 and $\bar{\mu}^2$ is the Ω-transform of a solid of A_5, in the former case of a solid T which meets φ in four distinct points $P_1, \ldots, P_4$, and in the latter case of a solid Σ which touches φ at a single‡ point P. If |G| and |F| denote the linear systems of surfaces, in T and Σ respectively, that represent prime sections of μ^2 and $\bar{\mu}^2$ respectively, then |G| is the *complete* system of cubic surfaces in T that pass through the points $P_1, \ldots, P_4$, and μ^2, accordingly, is a *normal* variety $V_3^{23}[15]$; but |F| is a sub-system (not necessarily complete) of the cubic surfaces in Σ that touch a fixed plane – the tangent plane τ^* to φ at P – at a fixed point P of this plane. If $|\bar{F}|$ denotes the *complete* system of such surfaces, then $|\bar{F}|$ is of freedom 16 and grade 23, so that its projective model is a normal threefold $H_3^{23}[16]$, whence, since $\bar{\mu}^2$ is a specialization of μ^2, we deduce that |F| must be a proper sub-system of $|\bar{F}|$. We find, in fact, by an inspection of the parametric equations of $\bar{\mu}^2$, that |F| is a sub-system of freedom 15 of $|\bar{F}|$; and hence $\bar{\mu}^2$ is a threefold $W = W_3^{23}[15]$, projection of $H_3^{23}[16]$ from a point U external to the latter. To study W, therefore, we have first to study in some detail the normal variety $H = H_3^{23}[16]$ and then to discuss the projection of H into W from a suitable external point U. While outlining, stage by

† For the record, it should be noted that we disagree with Severi's derivation [S, pp. 317–319] of a minimum base for surfaces on δ. Severi finds that such a minimum base consists of four surfaces which he denotes by π'_x, π'_g, G′, Ψ′, these being the surfaces which we have denoted by τ, α, σ and ρ respectively (the last two forming the base for surfaces on δη); and he attempts to prove that these four surfaces are algebraically independent on δ by showing that they have a non-singular self-intersection matrix on δ. However, our formula (11.4.4), which states that $\rho \sim 2\tau + 2\alpha$ on δ, already contradicts the independence of Severi's four surfaces on δ. Further, as regards Severi's intersection matrix, I find the following corrections to some of his intersection numbers:

$$[\pi'_x \cdot \Psi'] = 2, \quad [G' \cdot \Psi'] = 2 \quad \text{and} \quad [\Psi' \cdot \Psi'] = 4;$$

and these corrections do in fact result in an intersection matrix of determinant 0.

‡ A solid of A_5 that touches φ at a point, and has no other point in common with φ, is the intersection of a unique pair of contact primes of φ.

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stage, the course and results of this investigation, we do not consider it necessary to reproduce here the details of the lengthy algebraic computations that were required to carry it through.

Consider first then the normal variety H, projective model of the cubic surfaces in the solid T that touch the plane τ^* at the point P of this plane. To the first neighbourhood of P in T there corresponds a single point X of H; and we find that X is a quadruple point of H and that the nodal cone Ξ of H at X is a Veronese cone (having a Veronese surface as its general prime section). On the other hand we find that to the neighbourhood† in T of each direction d at P in τ^* there corresponds on H a line that passes through X and lies on Ξ; also that, as d varies (in τ^*) this line generates a quadric cone γ with X as vertex. Further, there exists on γ one particular conic $\bar{b}$ whose points correspond to sections of neighbourhoods of the directions d by the plane τ^*. To the plane τ^* itself there corresponds on H a rational quintic scroll $\bar{\tau}$ with $\bar{b}$ as directrix conic; and finally, to the quadric surface ω^* in which T meets M, residually to τ^*, there corresponds on H a surface $\bar{\omega}$ – an F^{14} (cf. §7.4) – which contains the conic $\bar{b}$ and meets $\bar{\tau}$ in two of its generators $\bar{g}_1, \bar{g}_2$.

As regards the point U from which H is to project into a threefold $W = \bar{\mu}^2$, we find that U, being external to both H and Ξ, lies in the ambient solid of the quadric cone γ but not in the plane of the conic $\bar{b}$. This implies, on projection from U, that W has a quadruple point which may be taken to be X; but it now possesses a double plane β through X, projection of the cone γ; and the conic $\bar{b}$ projects into a conic b in β (not through X). The nodal cone of W at X – projection of Ξ – is a special Veronese cone with β as double plane. Further, keeping to the notation of §7.5, the scroll $\bar{\tau}$ projects into a scroll τ with b as directrix conic, and the surface $\bar{\omega}$ projects into a surface ω that passes through b and meets τ in two of its generators g_1, g_2.

We now give the interpretation of the above results in terms of the mapping on W $(=\bar{\mu}^2)$ of the system of (complete) conics through two points A, B of S_2:

(i) X maps the η-conic with A, B as eyes,
(ii) b maps $\delta\eta$-conics with AB as axis,
(iii) β maps η-conics with AB as axis,
(iv) τ maps δ-conics with AB as one arm,
(v) ω maps δ-conics with arms through A and B respectively, and
(vi) g_1, g_2 represent δ-conics with AB as one arm and the other arm through A or B respectively.

The intersection $\bar{\mu}^2\delta$ consists of τ and ω (of combined order 19); $\bar{\mu}^2\eta$ consists of the plane β counted quadruply; and $\bar{\mu}^2\delta\eta$ is the conic b, also counted quadruply.

A 2. *The threefold* $\mu\nu$

This maps complete conics (S, E) such that S is apolar to a fixed conic-envelope σ while E is apolar to a fixed conic-locus s. Its table of virtual orders, analogous to that for μ^2 in §10, is

	$\mu\nu$	$\mu\nu\delta = \mu\nu\eta$	$\mu\nu\delta\eta$
virtual order	28	14	16

† Let $\bar{H}$ be the threefold $\bar{H}_3^{26}[18]$ which is the projective model of cubic surfaces of T that pass (simply) through P. Further let π be the plane of $\bar{H}$ that corresponds to the neighbourhood of P. Then H is the projection of H from a line t of π. The tangent solids to $\bar{H}$ at points of t form a quadric cone, and they project from t into the generators of a (two-dimensional) quadric cone on H whose vertex X is the projection of π. Further the plane τ^* in T represents a surface ${}^1F^8$ on $\bar{H}$, which contains t; and the tangent planes to this surface at points of t project into the points of a conic $\bar{b}$ on the above-mentioned quadric cone.

The δ-conics of the (self-dual) system are defined by the pairs of lines that are conjugate for σ and meet on *s*; the η-conics by the point-pairs that are conjugate for *s* whose joins touch σ; and the δη-conics are those whose vertices lie on *s* and whose axes touch σ. The surface μνδ is a rational scroll ${}^0R^{14}$ of which each generator maps δ-conics (of the system) with a fixed vertex on *s*; and μνη is a scroll ${}^0S^{14}$ whose generators correspond dually to the tangents to σ; and these two scrolls meet in an elliptic curve ${}^1C^{16}$ which is μνδη. Each of ${}^0R^{14}$ and ${}^0S^{14}$ is generated by chords joining point-pairs of a rational involution on ${}^1C^{16}$.

The A_5-model of μν is a quadric V_3^2, section of a quadric Φ by a prime Π; and μν is therefore the projective model of sections of V_3^2 by the cubic primals of Π that contain a rational normal quartic curve *c* of V_3^2, section of ϕ by Π. The neighbourhood of *c* on V_3^2 represents ${}^0S^{14}$; and ${}^0R^{14}$ is represented on V_3^2 by the sextic surface – generated by chords of *c* – in which V_3^2 is met by the chord locus M_3^2 of *c*.

A 3. *The threefold* $\overline{\mu\nu}$

This maps the (complete) conics of S_2 that pass through a fixed point P and touch a fixed line *p* not passing through P. We denote this system of conics by Σ. Thus Σ is a self-dual system, clearly a specialization of the system represented by μν; and $\overline{\mu\nu}$ must have the same table of virtual orders, properly interpreted, as that given for μν.

The δη-conics of Σ (axes through P, vertices on *p*) are mapped on a rational normal quartic curve *c* of Ω, a curve of the equivalence class $a+b$; its δ-conics (vertices on *p* and one arm through P) are found to be mapped on Ω by a rational septimic scroll ${}^0R^7$ with *c* as a directrix curve, and similarly its η-conics are mapped on another scroll ${}^0S^7$ with *c* as directrix curve. Each of these scrolls is the projective model of a plane curve system $C^4[O^3]$.

The A_5-model of Σ is a quadric line-cone $\mathcal{Q} = V_3^2$, section of a quadric Φ^* of A_5 (cf. §6) by a contact prime Π of ϕ, whose conic of contact with ϕ is *k*. Now Φ^* is the quadric plane-cone that projects ϕ from a tangent plane τ* of this surface; whence the vertex of $\mathcal{Q}$ is a line *v* in τ*, its generating planes join *v* to the points of *k*, it touches ϕ at each point of *k*, and touches the primal M along a cubic scroll γ which corresponds to ${}^0R^7$ on Ω. The Ω-model $\overline{\mu\nu}$ is the transform of $\mathcal{Q}$ by cubic primals that touch ϕ along *k*, and it is a V_3^{28}.

The table of virtual orders is to be interpreted by saying that $\overline{\mu\nu}\delta$ and $\overline{\mu\nu}\eta$ are the scrolls ${}^0R^7$ and ${}^0S^7$ respectively, each counted twice; while $\overline{\mu\nu}\delta\eta$ is the curve *c* counted quadruply.

A 3.1. *The threefold* $\widehat{\mu\nu}$

This maps, as we saw in §7.3, the system Σ of (complete) conics of S_2 that have a given point Q and a given line *q* as pole and polar. The A_5-model of Σ is a solid T that meets ϕ in a conic *k* and in a residual point P remote from *k*. We find then that $\widehat{\mu\nu}$ is the projective model of the complete system of cubic surfaces in T that pass through *k* and P, being therefore a $V_3^{14}[11]$. The table of virtual orders for $\widehat{\mu\nu}$ (remembering that $\mu\nu \equiv 2\widehat{\mu\nu}$) is

	$\widehat{\mu\nu}$	$\widehat{\mu\nu}\delta = \widehat{\mu\nu}\eta$	$\widehat{\mu\nu}\delta\eta$
virtual order	14	7	8

The plane π of *k* represents a plane α on $\widehat{\mu\nu}$, model of δ-conics with Q as vertex; and the neighbourhood of *k* in π represents a conic *a* in α, model of the δη-conics with Q as vertex. On the other hand, the neighbourhood of P in T represents a plane β on $\widehat{\mu\nu}$, model of η-conics with q as axis; and the neighbourhood of P on the quadric cone P(k) represents the conic b in β that maps $\delta\eta$-conics with q as axis.

Further, the quadric cone P(k) corresponds to a rational sextic scroll κ on $\widehat{\mu\nu}$ (cf. §7.5(ii)); and this maps a second system of δ-conics of Σ, namely those whose arms correspond in the harmonic homology of S_2 with Q as vertex and q as axis. Dually, Σ contains a second system of η-conics, those defined by corresponding point-pairs in the above homology; and these are mapped on a sextic scroll κ' (of the type dual to κ) on $\widehat{\mu\nu}$. The A_5-model of κ' is the neighbourhood of k in T. The neighbourhood of k on the cone P(k) corresponds to a rational normal quartic curve c on $\widehat{\mu\nu}$, the points of c mapping $\delta\eta$-conics of Σ with vertices on q and axes through Q. The surface κ has the conic b and the quartic curve c as simple directrix curves whereas κ' has the conic a and c as simple directrix curves.

In agreement with the table of virtual orders, $\widehat{\mu\nu}$ is a V_3^{14}; $\widehat{\mu\nu}\delta$ consists of α and κ; $\widehat{\mu\nu}\eta$ consists of β and κ'; and $\widehat{\mu\nu}\delta\eta$ consists of the three curves a, b and c, of combined order eight.

A 4. *The threefold* $\overline{\mu\nu}$

This threefold, in our notation, is the Ω-model of a system Σ of conics of S_2 that touch a given line q at a given point Q. It is, plainly, a proper specialization of $\widehat{\mu\nu}$ and will have the same table of virtual orders, with such multiplicity interpretations as may be necessary. The A_5-model of Σ is a solid T which meets ϕ in a conic k and touches ϕ at a point P of k (instead of meeting ϕ in a residual point remote from k). If τ^* is the tangent plane of ϕ at P, then we find that $\overline{\mu\nu}$ *is the projective model of the complete system of cubic surfaces in* T *that contain* k *and touch* τ^* *at the point* P *of* k. Further, if π is the plane of k, then the section of M by T consists of π counted twice, together with the plane τ^*.

Here again, since the degeneration sub-varieties of $\overline{\mu\nu}$ are complicated, we shall confine ourselves largely to a statement of results, as follows.

(i) *The* $\delta\eta$*-conic with* Q *as vertex and* q *as axis*. This is obviously a very special member of Σ. Its image on $\overline{\mu\nu}$ is the point X which corresponds to the neighbourhood of P in T. We find, then, that X is a triple point of $\overline{\mu\nu}$, and that the nodal cone Ξ of $\overline{\mu\nu}$ at X is such that its general prime section is a rational normal cubic scroll $^0R^3[4]$.

(ii) δ*-Conics with vertex* Q. These are mapped on a plane α of $\overline{\mu\nu}$ which lies on Ξ and corresponds to the conic-plane π of ϕ.

(iii) $\delta\eta$*-Conics with vertex* Q. These are mapped on the conic a in α; this passes through X and represents the neighbourhood of k in π.

(iv) δ*-Conics with* q *as one arm*. These are mapped on a rational quintic scroll τ, transform of the tangent plane τ^* of ϕ.

(v) δ*-Conics with* q *as one arm and the other arm through* Q. These are mapped on the tangent z_a to a at X; and z_a is also the generator of τ through X.

Dual to systems (ii), ..., (v) we have the following:

(vi) η*-Conics with axis* q. The map of these is a plane β, also through X and lying on Ξ. This corresponds to the 'tangential neighbourhood' of τ^* at P (cf. the case of $\bar{\mu}^2$).

(vii) $\delta\eta$*-Conics with axis* q. These are mapped on the conic b in β; and b, which meets a at X, is the directrix conic of τ.

(viii) η*-Conics with* Q *as one eye*. These are mapped on a second rational quintic scroll τ' with a as directrix conic.

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(ix) η-*Conics with Q as one eye, and the other eye on q*. These are mapped onto the tangent z_b to b at X; and z_b is the generator of τ' through X.

The intersection $\bar{\mu}\nu\delta$ consists of α counted twice together with the quintic scroll τ; and there is a similar result for $\bar{\mu}\nu\eta$. Also $\bar{\mu}\nu\delta\eta$ consists of the pair of conics a, b each counted twice.

Appendix B. Note on Halphen conditions

B 1. *Preliminary*

The early work of Halphen, Zeuthen and others on the foundations of enumerative calculus for complete conics has been reviewed with detailed references in Severi (1916), particularly in §10 of this work: and an account of Zeuthen's work appears in Zeuthen (1914, pp. 309–341). From this early work there emerged the particular importance of a special type of condition that could be imposed on complete conics. This is the type of condition – here to be called a *Halphen* condition – which is such that it is satisfied either by every $\delta\eta$-conic or by an abnormal plurality of $\delta\eta$-conics. In terms of condition manifolds on Ω, this means that the condition manifold of a Halphen condition either contains the threefold $\delta\eta$ on Ω entirely or meets this threefold in a sub-variety of abnormal dimension.† The use of such conditions leads obviously to situations in which a set of conditions, of combined weight five, are satisfied by an infinity of $\delta\eta$-conics; and in which, therefore, such basic enumerative formulae as (7.1.3) are inapplicable. This in turn leads to a new and more difficult type of enumerative problem, here to be called a *Halphen* problem, in which it is sought to compute the number of complete conics, *other than* $\delta\eta$-*conics*, that satisfy a set of conditions, of combined weight five, when these are already satisfied by an infinite family of $\delta\eta$-conics, supposing only that the number of the residual solutions involved is finite. It is precisely with this type of problem that Severi is concerned (see **S**, §10 and pp. 292–301), and we now wish to refer briefly to his approach and to the results that he obtains. The basis of his method is consideration of geometry – including relations of equivalence – on the open variety $\Omega-\delta\eta$.

B 2. *Geometry on* $\Omega-\delta\eta$

By the removal of $\delta\eta$ from Ω, it is clear that all fourfolds on Ω, as also all curves on Ω that meet $\delta\eta$, are reduced to open varieties. But, where no ambiguity is possible, we shall use the same symbols to denote these open varieties as we have previously used to denote their closures on Ω.

Severi's first objective then is to find a base for the equivalence of fourfolds on $\Omega-\delta\eta$; and his result is that the triad of varieties (μ, ν, δ) constitutes such a base. The steps by which he arrives at this conclusion can be summarized briefly as follows. He first notes that, in the birational representation of Ω on the space A_5 of conic loci, any fourfold on Ω that passes (simply or multiply) through $\delta\eta$ corresponds to a primal of A_5 that passes at least doubly through the base Veronese surface ϕ in A_5 and has nodal contact over ϕ with the chord-locus M of ϕ (its nodal cone at each point of ϕ having the nodal cone of M at this point as a component). Using this, he then finds an equivalence formula, *valid equally on* Ω *and on* $\Omega-\delta\eta$, which exhibits every fourfold (either on Ω or on $\Omega-\delta\eta$) as a linear combination of μ, ν and δ. On Ω, where μ, ν and δ are dependent – by virtue of the equivalence relation $\delta \sim 2\nu-\mu$ – the above result only expresses that (μ, ν)

† Plainly each of the conditions δ and η is a Halphen condition, as is also any simple condition whose condition manifold is the section of Ω by a primal that contains the threefold $\delta\eta$.

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is a base for fourfolds on Ω; but on $\Omega - \delta\eta$, where the fourfolds μ, ν and δ are not so related, the result means that μ, ν and δ are a base for fourfolds on $\Omega - \delta\eta$.

Next, as regards a base for curves on $\Omega - \delta\eta$, Severi first considers the following families of curves on Ω:

(i) the ∞^8-family of curves a – twisted cubic curves trisecant to δ – which represent the ordinary pencils of conics of S_2 (the lines of A_5);

(ii) the ∞^6-family of conics l, each representing a pencil-range of conics of S_2, such that, on Ω, $l\delta = l\eta = 1$, while l does not meet $\delta\eta$; and

(iii) the ∞^5-family of conics t, each representing a hyperosculating (four-point contact) pencil-range of S_2, and each meeting $\delta\eta$ in one point which thus replaces the separate points (on Ω) in which a conic l meets δ and η respectively. In the notation of §§7.1 and 7.2, we see that $a \sim u+2v$, $l \sim u+v$, while t (on Ω) is a proper specialization of l; so that (a, l) is a legitimate choice of base for curves on Ω. On $\Omega - \delta\eta$, however, the intersection of t with $\delta\eta$ has been removed; so that, whereas $l\delta = l\eta = 1$, we have $t\delta = t\eta = 0$. Thus t is not equivalent to l on $\Omega - \delta\eta$; and Severi takes the triad (a, l, t) as a base for curves on $\Omega - \delta\eta$, complementary to the base (μ, ν, δ) for fourfolds.

Severi then computes that (on $\Omega - \delta\eta$)

$$[\mu a] = [\mu l] = [\mu t] = 1,$$

$$[\nu a] = 2, \quad [\nu l] = [\nu t] = 1,$$

$$[\delta a] = 3, \quad [\delta l] = 1, \quad [\delta t] = 0,$$

whence the relevant intersection matrix for the bases (μ, ν, δ) and (a, l, t) on $\Omega - \delta\eta$ is

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \end{bmatrix}$$

which has determinant 1. From this Severi infers that (μ, ν, δ) *and* (a, l, t) *are minimum bases for fourfolds and curves on* $\Omega - \delta\eta$.

Translating the above result about curves on $\Omega - \delta\eta$ into the language of quadruple conditions on complete conics, Severi defines the following *characteristics* of the totality of quadruple conditions on complete conics relative to the geometry of $\Omega - \delta\eta$:

μ': the condition on a complete conic that its image point on Ω should lie on a given curve a, i.e. that the complete conic should belong to a given general pencil of conics;

ν': the condition on a complete conic that its image point should lie on a given conic l, i.e. that the conic belong to a given general pencil-range; and

τ': the condition on a complete conic that its image point lie on a given conic t, i.e. that the complete conic belong to a given hyperosculating pencil-range.

With the use of this notation, Severi now computes the values of the coefficients α, β, γ when a given simple condition c is expressed (on $\Omega - \delta\eta$) in the form $c \sim \alpha\mu + \beta\nu + \gamma\delta$ (α, β, γ integers). Thus he finds that

$$c\mu' = \alpha + 2\beta + 3\gamma, \quad c\nu' = \alpha + \beta + \gamma \quad \text{and} \quad c\tau' = \alpha + \beta,$$

whence it follows that

$$\alpha = c(3\nu' - \mu' - \tau'), \quad \beta = c(\mu' + 2\tau' - 3\nu') \quad \text{and} \quad \gamma = c(\nu' - \tau').$$

In particular he finds that $$\eta \sim 3\mu - 3\nu + \delta,$$

a formula which reduces on Ω to the known formula $\eta \sim 2\mu - \nu$. Similarly, Severi computes the values of α', β', γ' when a fourfold condition on complete conics is written in the form

$$\alpha'\mu' + \beta'\nu' + \gamma'\tau'.$$

B 3. *Some further remarks on Halphen problems*

In a short note, arising from Severi's theory as outlined above, the present author (Semple 1951) made some further suggestions. In the first place, he suggested that instead of adjoining δ to μ, ν to give a base for fourfolds on $\Omega - \delta\eta$, the same effect (with more symmetry) could be achieved by adjoining to μ, ν any general fourfold ρ which is the section of Ω by a quadric through $\delta\eta$. On Ω, in fact, we have $\rho \sim 2(\mu+\nu)$, so that $\rho \sim 3\mu + \delta$. A particular concrete example of a condition ρ is as follows:

The fourfold system of complete conics of S_2 *that have four-point contact with some conic of a given self-polar net* (*the net of conics with a given self-polar triangle*) *is a Halphen system of conics that contains all* $\delta\eta$*-conics.*

If the equation of the self-polar net in S_2 is taken to be $\lambda_0 x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2 = 0$, and if the general conic of S_2 is taken to be $x^T A x = 0$, then the above condition on this conic can be written in the form

$$A_{12}A_{20}A_{01} + a_{12}a_{20}a_{01}\,|A| = 0$$

which, since $|A| = 0$ is the equation of M in A_5, and since $A_{12} = 0$, $A_{20} = 0$, $A_{01} = 0$ are the equations of three quadrics through ϕ, represents in A_5 a sextic primal passing doubly through ϕ and having nodal contact with M over ϕ; and such primals (see § 2) correspond to sections of Ω by quadrics through $\delta\eta$.

It was next remarked (by use of the A_5-representation), that the total intersection of δ with a pair ρ_1, ρ_2 of the sections of Ω by quadrics through $\delta\eta$ consists of the threefold $\delta\eta$ together with a surface consisting of a finite number of the planes α that generate δ; and it was found that the number of these planes is 36. By use of this result and further degeneration arguments, it was then found possible to compute all the numbers

$$\mu^\alpha\nu^\beta\rho^\gamma \quad (\alpha+\beta+\gamma = 5, \gamma > 1),$$

in which the factor ρ^γ $(\gamma > 1)$ is to be interpreted as the intersection, residual to $\delta\eta$, of γ of the fourfolds ρ. Thus, for example, it was found that $\rho^5 = 1296$. The numbers $\mu^\alpha\nu^\beta\rho^\gamma$ can now be used for a limited class of calculations involving simple Halphen conditions of such a kind (as we may say) as are sufficiently characterized by their representations in the form $a\mu + b\nu + c\rho$ (a, b, c integers).

At the end of this note, the suggestion is made that another mode of attack on the same problems could be to dilate $\delta\eta$ into a fourfold, $\bar{\varepsilon}$ say, on a birational model $\bar{\Omega}$ of Ω. Thus, for example, $\bar{\Omega}$ might be defined as the projective model of all sections of Ω by *cubic* primals through $\delta\eta$. (Quadric primals through $\delta\eta$ would have all the conic planes of $\delta\eta$ as fundamental planes.) We shall not attempt here, however, any further examination of this suggestion.

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